Hamiltonian decomposition of prisms over cubic graphs

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The prisms over cubic graphs are 4-regular graphs. The prisms over 3-connected cubic graphs are Hamiltonian. In 1986 Brian Alspach and Moshe Rosenfeld conjectured that these prisms are Hamiltonian decomposable.
In this paper we present a short survey of the status of this conjecture, various constructions proving that certain families of prisms over 3-connected cubic graphs are Hamiltonian decomposable. Among others, we prove that the prisms over cubic Halin graphs, cubic generalized Halin graphs of order $4k + 2$ and other infinite sequences of cubic graphs are Hamiltonian decomposable.

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1 Introduction

Definition 1 *The* **prism** *over a graph G is the Cartesian product $G \times K_2$. In other words, we take two copies of G,* **upper** *copy and* **lower** *copy, and join each vertex to its clone in the other copy by a* **vertical** *edge. (See Figure 1).*

Remark 2 *It is easy to see that the prism over a 2-connected cubic graph is a 4-connected 4-regular graph.*

Definition 3 *A* **Hamiltonian decomposition** *of a graph is a partition of its edges such that each part induces a Hamiltonian cycle. A graph is* **Hamiltonian decomposable** *if it admits a Hamiltonian decomposition. A graph is* **prism decomposable** *if the prism over it is Hamiltonian decomposable.*

Throughout this paper, we use standard notation and definitions for graphs as in [2] or any other book on graph theory. Our study of prisms over graphs was motivated by D. Barnette's still open conjecture

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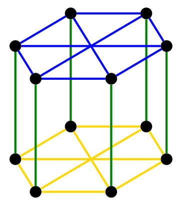


Fig. 1: The prism over $K_{3,3}$.

(1970) (see [5, page 1145]) that all simple 4-polytopes [(i)] are Hamiltonian. This conjecture was probably motivated by Tutte's remarkable and surprising theorem (see [10]) that all 4-connected 3-polytopes are Hamiltonian. It is a remarkable result as these graphs are sparse, at most $3n-6$ edges in a graph of order n; the prisms over sparse graphs are also sparse. The simplicity requirement in Barnette's conjecture is essential as it is easy to construct non-Hamiltonian 4-polyhedral graphs.

In 1973, we tested this conjecture on prisms over simple 3-polytopes which are simple 4-polytopes (see [8]). We observed that the 4-color conjecture (which became a theorem in 1976) implies that these prisms are Hamiltonian. This paper introduced the "***B-Y spanning subgraph***" which was later used to prove that the prism over 3-connected cubic graphs (even non-planar) are Hamiltonian (see [7, 3]). In 1986, together with Brian Alspah, we observed that the prisms over all 3-connected cubic graphs we tested actually were Hamiltonian decomposable. In [1] we conjectured that:

Conjecture 4 *The prisms over 3-connected cubic graphs are Hamiltonian decomposable.*

In 2008 A. Bondy and U. S. R. Murty, in their book [2] wrote: *" We present here an updated selection of interesting unsolved problems and conjectures"*. This conjecture is listed as problem $\#85$ in this book.

The conjecture has been verified for 3-connected cubic bipartite planar graphs, for the duals of Kleetops [(ii)] (see [3]), for prisms ($C_k \times K_2$), for 3-edge-colorable cubic graphs such that every two colored 1-factors form a Hamiltonian cycle (such as K_4 or the Dodecahedron see [1]).

The 3-connectivity is essential as it is possible to find 2-connected cubic graphs which are not prism decomposable (see [3]). As an aside, prisms over 2-connected planar graphs are Hamiltonian (see [4]). Probably there are 2-connected cubic graphs whose prisms are not Hamiltonian, but so far they have alluded us.

In this note, we exhibit a variety of constructions of Hamiltonian decompositions of prisms over cubic graphs. Some of these constructions apply to certain cubic graphs while they do not apply to others. This led us to conclude that it is unlikely that one will be able to find a single proof for Conjecture 4, assuming of course that it is true, which we believe.

1.1 Preliminaries

Given a cubic graph G, if the prism over G has a Hamiltonian cycle C, then the edges of G can be partitioned into four sets:

- The edges of G that appear only in the upper part of C (we will refer to them as the blue edges);

[(i)] All simple 4 polytopes are 4-connected 4-regular graphs, but not every such graph is a graph of a 4-polytope.
[(ii)] Cubic graphs that are obtained by starting from K_4 and repeatedly "inflating" vertices to triangles.

- The edges of G that appear only in the lower part of C (the yellow edges);
- The edges of G that appear in both parts. (green edges, blue and yellow combined);
- The edges of G that are not used in C.

It was observed in [8] that the blue-yellow edges trace in G a set of (vertex) disjoint blue-yellow colored even cycles, and the green edges form a disjoint collection of paths. The union of these edges is a spanning connected sub-cubic subgraph of G. We call this colored subgraph a ***B-Y subgraph***.

An example of a B-Y subgraph is the even cactus (see Figure 2). The proofs that the prisms over 3-connected cubic graphs are Hamiltonian were accomplished in two steps: first they have a spanning 2-connected bipartite sub-cubic subgraph, and second every such subgraph has a spanning even cactus subgraph. In general, for the prism over a graph G to be Hamiltonian, it is not necessary to have a spanning 2-connected bipartite sub-cubic subgraph, for example, the Kleetops.

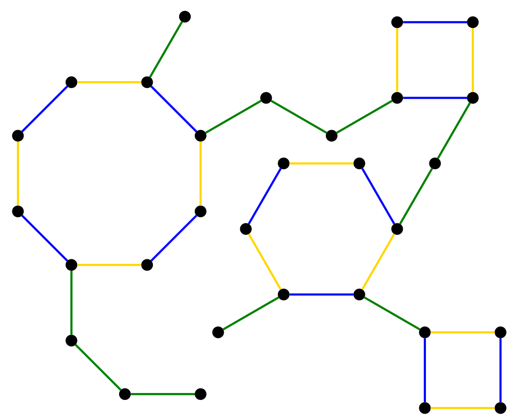


Fig. 2: An even cactus.

Remark 5 *Throughout the paper, we color the edges of the cubic graph G as described in §1.1. When we construct two B-Y subgraphs which share the same B-Y cycles, in order to distinguish two sets of green edges in the two subgraphs, we color green the edges in one copy and red in the other. See Figure 3(a) for an example.*

Given a spanning B-Y subgraph of a cubic graph G, in order to trace the Hamiltonian cycle it represents in the prism, start at any vertex v_0. Select a colored edge (say blue) proceed along the edge to the neighbor v_1. If there is a green edge, then proceed along it, if not, use the vertical edge and follow the yellow edge on the lower copy (recall that a green edge means that the edge is both blue and yellow). Repeat this procedure until we get back to v_0, the starting vertex. See Figure 3 for an example.

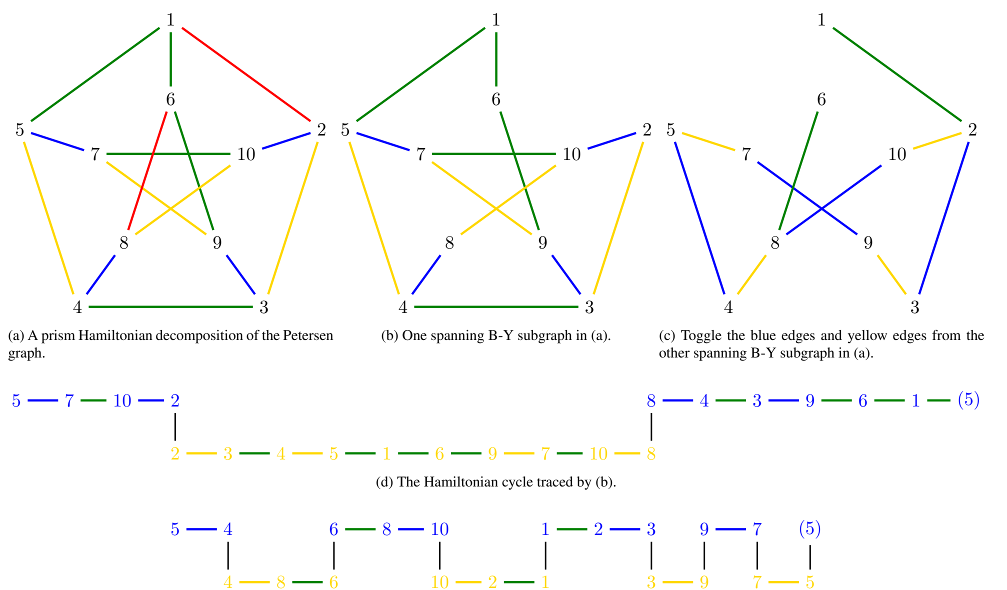
The following theorem was, and still is the main tool for proving that a cubic graph G is prism decomposable:

Theorem 6 *[1, Theorem 3] A cubic graph is prism decomposable if and only if there exists two spanning B-Y subgraphs of G such that:*

1. *The two B-Y spanning subgraphs share the same B-Y cycles;*

2. *Each edge in G other than the ones in the common B-Y cycles belongs to exactly one B-Y spanning subgraph.*

We call the two spanning B-Y subgraphs that satisfy the two conditions in Theorem 6 a ***prism Hamiltonian decomposition***. From now on, in order to prove that a cubic graph is prism decomposable, instead of showing two concrete edge-disjoint Hamiltonian cycles, we construct a prism Hamiltonian decomposition.



(a) A prism Hamiltonian decomposition of the Petersen graph.

(b) One spanning B-Y subgraph in (a).

(c) Toggle the blue edges and yellow edges from the other spanning B-Y subgraph in (a).

(d) The Hamiltonian cycle traced by (b).

(e) The Hamiltonian cycle traced by (c).

Fig. 3: The Petersen graph is prism decomposable. Figures (d), (e) are the two edge-disjoint Hamiltonian cycles of the prism over the Petersen graph traced by the two spanning B-Y subgraphs shown in (a). The color usage is described in Remark 5.

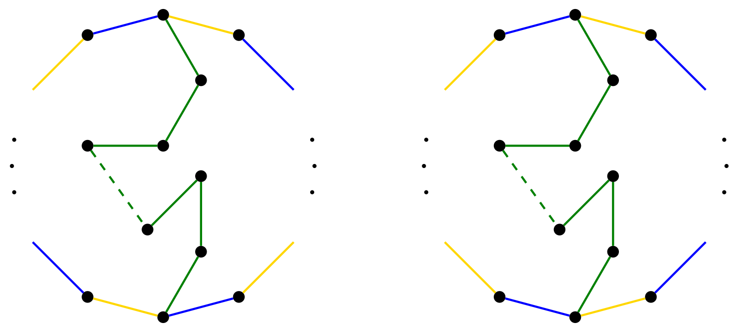
Traditionally, when we have a conjecture related to cubic graphs, we test it first on the Petersen graph. Figure 3 shows that the Petersen graph is prism decomposable and also demonstrates the use of prism Hamiltonian decomposition.

1.2 Gadgets

In this section we introduce gadgets, a collection of sub-cubic graphs that will help us put together the two spanning B-Y subgraphs.

Definition 7 *By a* **gadget** *we mean a connected sub-cubic graph with edges colored blue, yellow and green such that the blue-yellow edges form a set of disjoint even cycles, the green edges form a disjoint* *set of paths and the prism over this graph is Hamiltonian. We also require that the Hamiltonian cycle uses the vertical edges at all vertices of degree* 1*, or degree* 2 *where the two edges sharing this vertex are colored blue and yellow.*

It is easy to see that even cycles are gadgets.



(a) An odd diagonal.

(b) An even diagonal.

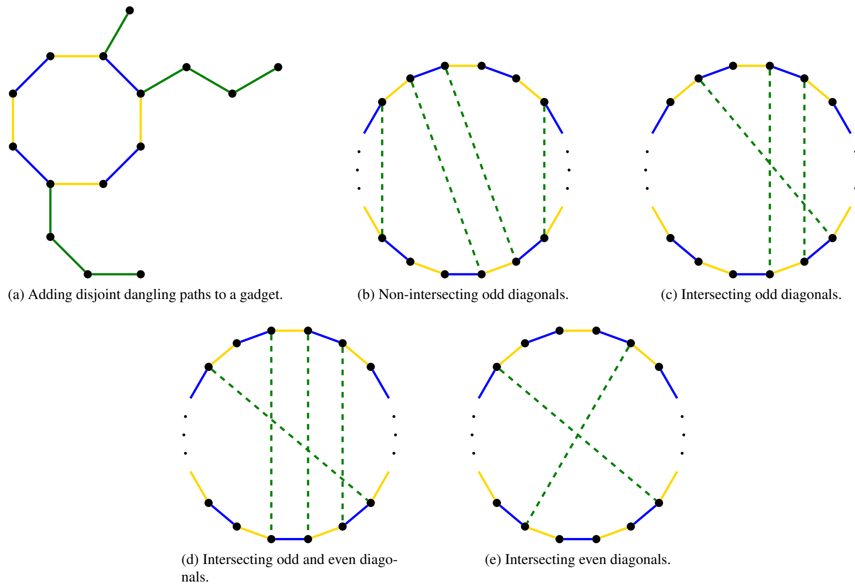
Fig. 4: Two kinds of diagonals in a B-Y cycle.

Definition 8 *Let* u, v *be two vertices on an even cycle* C*. A* **diagonal** *connecting* u *and* v *in* C*, denoted by* u—v*, is a path connecting* u *and* v *which contains no other vertices of the cycle* C*. A diagonal* u—v *is an* **odd** *diagonal if* u *and* v *have even distance along* C*. See Figure 4(a). Similarly, a diagonal* u—v *is an* **even** *diagonal if* u *and* v *have odd distance along* C*. See Figure 4(b). Two diagonals* a—b *and* c—d *are* **intersecting** *if they are disjoint and* a, c, b, d *appear in this order on the cycle* C*. See Figure 5.*

The following is a list of a few gadgets and methods to combine them in order to form new gadgets.

- G1: An even cycle.
- G2: Given a gadget, adding disjoint dangling green paths hanging from degree 2 vertices incident with blue and yellow edges. See Figure 5(a).
- G3: An even cycle with disjoint non-intersecting odd diagonals. See Figure 5(b).
- G4: An even cycle with an even number of disjoint non-intersecting odd diagonals and one additional odd diagonal that intersects all of them. See Figure 5(c).
- G5: An even cycle with an odd number of disjoint non-intersecting odd diagonals and one additional even diagonal that intersects all of them. See Figure 5(d).
- G6: An even cycle with two intersecting even diagonals. See Figure 5(e).
- G7: Given a gadget, adding a C_4 by splitting a green edge as shown in Figure 6.
- G8: Given two disjoint gadgets, joining two vertices, each on a different gadget, each of degree 2 and each the end vertex of a blue edge, by a green path. See Figure 2.

All listed gadgets are easily verifiable. We leave this simple task to the reader.



(a) Adding disjoint dangling paths to a gadget. (b) Non-intersecting odd diagonals. (c) Intersecting odd diagonals.

(d) Intersecting odd and even diagonals. (e) Intersecting even diagonals.

Fig. 5: Dashed lines are disjoint diagonals.

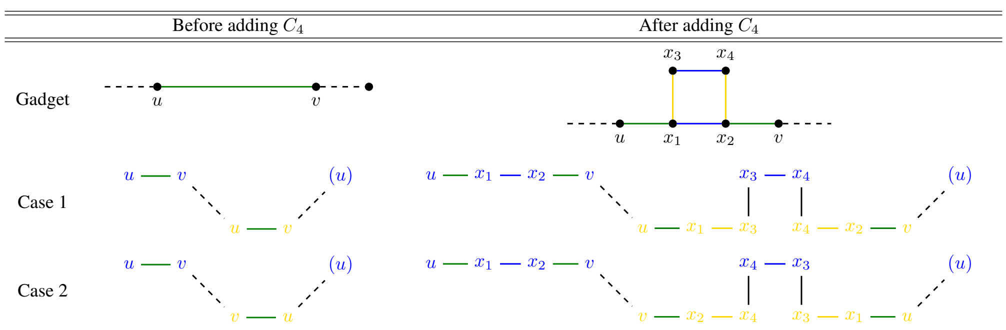


Fig. 6: Suppose we have a gadget in which uv is a green edge. By adding a C_4 between u and v, we obtain a new gadget.
Explanation: assume that we orient the Hamiltonian cycle so that the green edge uv is traversed in the upper level (blue) from $u \to v$. Case 1 describes the new Hamiltonian cycle in case the edge uv on the bottom level (yellow) is also traversed from $u \to v$, while case 2 describes the Hamiltonian cycle in case it is traversed from $v \to u$.

2 Prism decomposable cubic graphs

In this section we will demonstrate various techniques to prove that certain 3-connected cubic graphs are prism decomposable. We begin with an example of 2-connected cubic graphs (see Figure 7) that are not prism decomposable (see [3]). It is noteworthy that they demonstrate another subtle point. Nash-Williams conjectured that all 4-connected 4-regular graphs are Hamiltonian decomposable. Meredith showed us how to construct infinitely many counterexamples. The graphs we construct are Hamiltonian 4-connected 4-regular but not Hamiltonian decomposable.

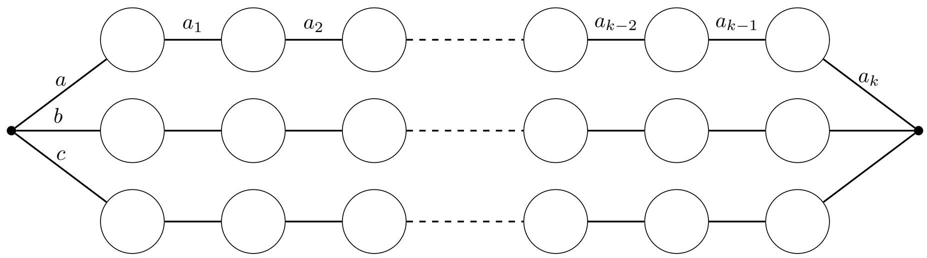
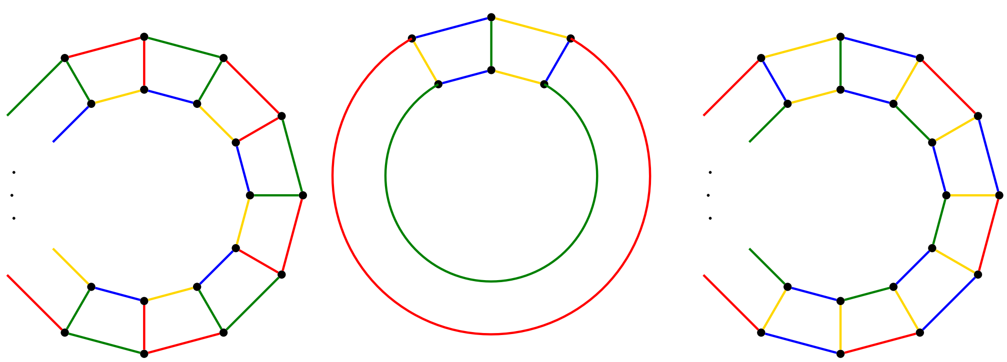


Fig. 7: Each circle stands for a 2-connected cubic graph with one edge deleted and the two resulting pair of vertices connected along the path $a, a_1, \ldots a_k$ and similarly the paths starting at b and c. Prisms over such graphs are 4-connected 4-regular (see [3]). They may be Hamiltonian but definitely not prism decomposable.

The following proposition was proved in [1]. We include it here in order to demonstrate the use of the gadgets.

Proposition 9 *The prisms $Pr_n := C_n \times K_2$ are prism decomposable.*

Proof: When n is even, $C_n \times K_2$ is planar and bipartite. We can also quickly demonstrate a direct construction of the two Y-B spanning subgraphs. Figure 8(a) shows a decomposition into two gadgets of type G2 over gadgets of type G1.



(a) A prism Hamiltonian decomposition of Pr_{2k}. (b) A prism Hamiltonian decomposition of Pr_3. (c) A prism Hamiltonian decomposition of Pr_{2k+1}.

Fig. 8: The prisms are prism decomposable.

When n is odd we resort to another approach. Figure 8(b) shows a prism Hamiltonian decomposition of the prism Pr_3. We can now use the gadget of type G7 to add $k-1$ copies of C_4 to obtain a prism

Hamiltonian decomposition of the prism Pr_{2k+1} as shown in Figure 8(c). □

We next apply our gadgets to prove that the generalized Petersen graphs are prism decomposable.

Definition 10 *A* **generalized Petersen** *graph is a cubic graph G that has an induced cycle C (called* **generalized Petersen cycle***) and an induced 2-regular graph H such that $V(C)$ and $V(H)$ give a partition of $V(G)$. Note that, the edges between $V(C)$ and $V(H)$ form a perfect matching.*

Proposition 11 *Every generalized Petersen graph G of order $4k$ is prism decomposable.*

Proof: Let C be a generalized Petersen cycle of order $2k$ in G. The graph $G \setminus V(C)$ is a union of disjoint cycles $A_1, \ldots, A_m$ where $A_i = a_{i,1}a_{i,2}\ldots a_{i,|A_i|}$. We color the edges of C by blue and yellow alternatingly. Let $\pi(a_{i,t})$ be the vertex of the cycle C matched to $a_{i,t}$ in G. We are now ready to construct two spanning B-Y subgraphs of G.

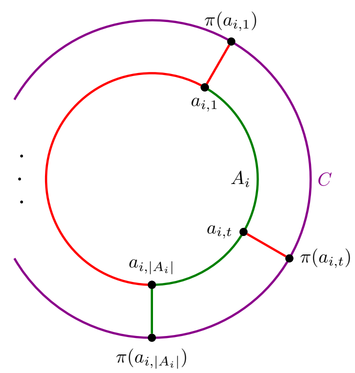


Fig. 9: Part of a prism Hamiltonian decomposition of a generalized Petersen graph of order $4k$. The purple cycle C is a B-Y cycle.

For each cycle A_i we color the edges along the path $a_{i,1}a_{i,2}\ldots a_{i,|A_i|}\pi(a_{i,|A_i|})$ green. Clearly, we get a single B-Y cycle plus dangling green paths, which are a type G2 gadget over a type G1 gadget, hence a spanning B-Y subgraph of G.

To form the second spanning B-Y subgraph, we color red the edges $a_{i,1}a_{i,|A_i|}$ and $\pi(a_{i,t})a_{i,t}$, $1 \leq t < |A_i|$. It is easy to see that the cycle C plus the dangling red paths are again a type G2 gadget over a type G1 gadget, hence a spanning B-Y subgraph of G. □

Definition 12 *A* **Halin graph** *is a planar graph consisting of a tree with no vertices of degree 2, embedded in the plane, plus a cycle (called the* **Halin cycle***) through its leaves connecting them in the order they are drawn in the plane.*

Definition 13 *A* **generalized Halin graph** *is a tree, with no vertices of degree 2, plus a cycle (called the* **generalized Halin cycle***) through the leaves.*

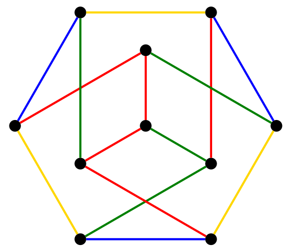


Fig. 10: The Petersen graph is a generalized Halin graph. A prism Hamiltonian decomposition using a single B-Y cycle (the generalized Halin cycle) is indicated.

In [6] it was proved that the prisms over generalized Halin graphs are Hamiltonian. A cubic generalized Halin graph of order $2k$ consists of a cycle C_{k+1} plus a tree with $k-1$ non-leaves (vertices of degree 3 in the tree). We believe that the prisms over cubic generalized Halin graphs are prism decomposable. We can only prove it for a bit more than half of them.

Before embarking on the proof we need the following Lemma:

Lemma 14 *Let T be a cubic tree (all non-leaves have degree 3). Given any two distinct leaves $u, v \in V(T)$, the edges of the tree can be partitioned into two disjoint sets $A^T_{u,v}$, $B^T_{u,v}$ satisfying:*

- *The set $A^T_{u,v}$ contains the edges of the unique path connecting u and v.*
- *Each set induces vertex disjoint paths that cover all non-leaves.*
- *For each set, every path covers exactly one leaf, except for the one path containing u and v.*

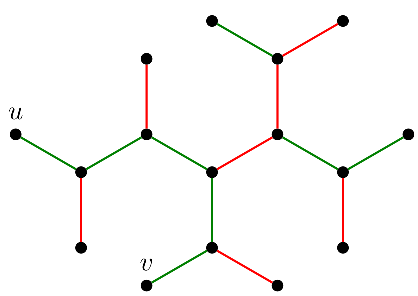


Fig. 11: A partition of the edges of the tree T in Lemma 14. The paths in $A^T_{u,v}$ are colored green, and those in $B^T_{u,v}$ are colored red.

Proof: By induction on the size of T. The result holds for K_1, K_2. Let T be a cubic tree of size at least 4. Pick a cherry, namely two different leaves x, y with a common neighbor z. Delete x, y to obtain a cubic tree T' in which z is a leaf. By the induction hypothesis, $E(T')$ has the required partition for every two distinct leaves in T'.

Case	$A^T_{u,v}$	$B^T_{u,v}$
$\{u,v\} \cap \{x,y\} = \emptyset$	$A^{T'}_{u,v} \cup \{xz\}$	$B^{T'}_{u,v} \cup \{yz\}$
$u = x, v \neq y$	$A^{T'}_{v,z} \cup \{xz\}$	$B^{T'}_{v,z} \cup \{yz\}$
$\{u,v\} = \{x,y\}$ ($z' \neq z$ is a leaf in T')	$B^{T'}_{z,z'} \cup \{xz, zy\}$	$A^{T'}_{z,z'}$

There are essentially three cases, and we list the decomposition for each case in the above table. □

Proposition 15 *The generalized Halin graph G of order $4k+2$ is prism decomposable.*

Proof: The generalized Halin cycle C is of order $2k+2$, hence an even cycle. Choose two different vertices u, v on the cycle C whose distance along the cycle is even. Let $T = G \setminus E(C)$, and $A^T_{u,v}$, $B^T_{u,v}$ be the decomposition obtained by Lemma 14. The set $E(C) \cup A^T_{u,v}$ is a type G2 gadget over a type G3 gadget. The set $E(C) \cup B^T_{u,v}$ is a type G2 gadget over a type G1 gadget. □

Proposition 16 *Every cubic generalized Halin graph G containing a vertex induced C_3 is prism decomposable.*

Proof: If G is of order $4k+2$, G is prism decomposable by Proposition 15. Now, assume that the graph G is of order $4k$. Let C be a generalized Halin cycle, which is of order $2k+1$. A $C_3 = \{a,b,c\}$ in G must contain two adjacent vertices in C, say a, b. Construct an even cycle C' by setting $E(C') = E(C) \triangle E(C_3)$, where $\triangle$ stands for the symmetric difference. Note that, in C', the diagonal a—b is an odd diagonal. Let $T = G \setminus E(C') \setminus E(C_3)$, which is a tree. From T, pick a leaf d which is not c. Applying Lemma 14, we obtain a decomposition $A^T_{c,d}$, $B^T_{c,d}$.

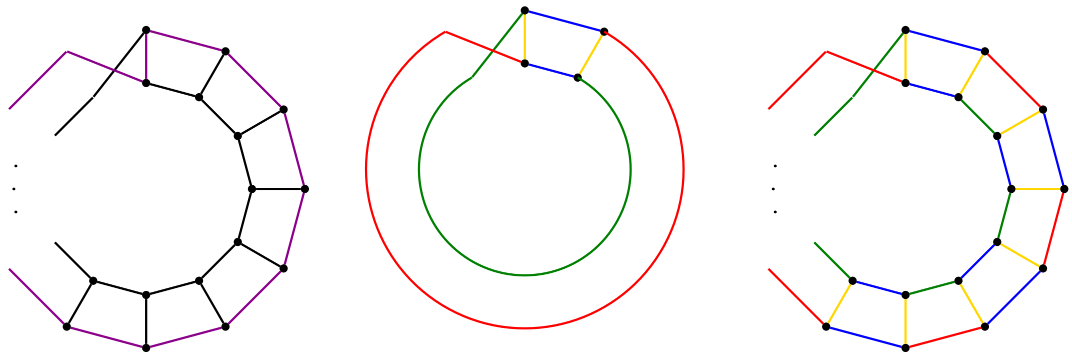
If the diagonal in C', c—d, is an odd diagonal, then $E(C') \cup A^T_{c,d}$ and $E(C') \cup B^T_{c,d} \cup \{ab\}$ are type G2 gadgets over type G3 gadgets. If the diagonal c—d is an even diagonal, then $E(C') \cup A^T_{c,d} \cup \{ab\}$ is a type G2 gadget over a type G5 gadget, and $E(C') \cup B^T_{c,d}$ is a type G2 gadget over a type G1 gadget. □

Corollary 17 *Cubic Halin graphs are prism decomposable.*

Proof: Every Halin graph contains a C_3, actually it is almost pancyclic, missing at most one cycle of even length (see [9]). The result follows from Proposition 16. □

The difficulty of finding prism Hamiltonian decompositions arises when we are short of "good" even cycles, that is even cycles of length near $\frac{|V(G)|}{2}$ that have very few single edge diagonals. For cubic generalized Halin graphs with an odd length generalized Halin cycle, we know that every two adjacent vertices on the cycle are connected by a unique path on the tree, which, together with the edge connecting the two vertices, form a cycle. We also know that at least one of these cycles is of odd length. When we combine it with the generalized Halin cycle, we obtain an even cycle. Sometimes, based on this single even cycle, we can construct the prism Hamiltonian decomposition.

But in general, a single even cycle is not enough. Actually, we can prove that, if a cubic graph G admits a prism Hamiltonian decomposition based on a single B-Y cycle C, $G \setminus E(C)$ must be a union of trees and connected unicycle graphs. Two extreme cases are the odd prisms Pr_{2k+1} (see Figure 8) and the Möbius ladders of order $4k$ (see Figure 12). While they are prism decomposable, it is not difficult to prove that, their prism Hamiltonian decompositions must involve multiple B-Y cycles.



(a) Every Möbius ladders is a generalized Halin graph. A generalized Halin cycle is colored in purple.

(b) A prism Hamiltonian decomposition of the Möbius ladder $M_4 \cong K_4$.

(c) A prism Hamiltonian decomposition of the Möbius ladder M_{4k}.

Fig. 12: Möbius ladders are prism decomposable.

Proposition 18 *The Möbius ladders M_n (see Figure 12) are prism Hamiltonian decomposable.*

Proof: Möbius ladders are generalized Halin graphs (see Figure 12(a)). Proposition 15 takes care of the graphs M_{4k+2}. In Figures 12(b) and 12(c), we employ the same strategy we used in Proposition 9 to prove that M_{4k} is prism decomposable. $\square$

We note that the Möbius graph is a graph based on a caterpillar and a cycle through its leaves (see Figure 12(a)). Such graphs may present the biggest challenge when we try to increase the size of the generalized Halin cycle to obtain an even cycle, as it will generate many single edge diagonals in the cycle. It is conceivable that proving that all cubic generalized Halin graphs based on a caterpillar are prism decomposable may lead to a general proof.

We conclude this section with an example of such a graph where we have many different prism Hamiltonian decompositions.

Proposition 19 *The Halin graph in Figure 13 has exponentially many prism Hamiltonian decompositions.*

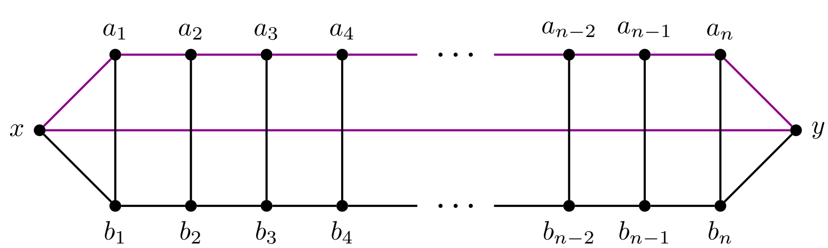
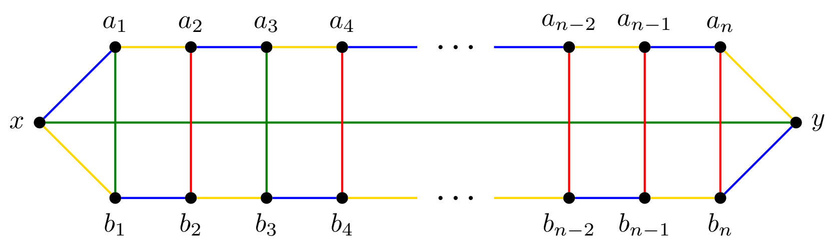
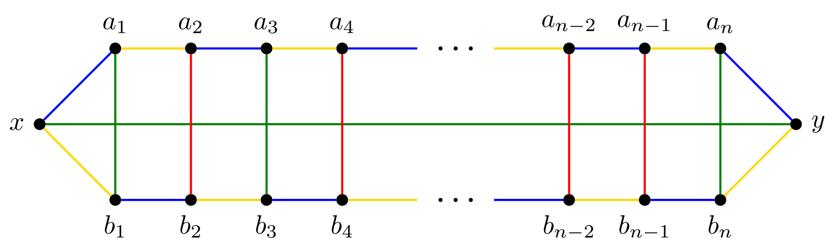


Fig. 13: A cubic Halin graph of order $2n+2$, consisting of a Halin cycle (colored purple) and a caterpillar.

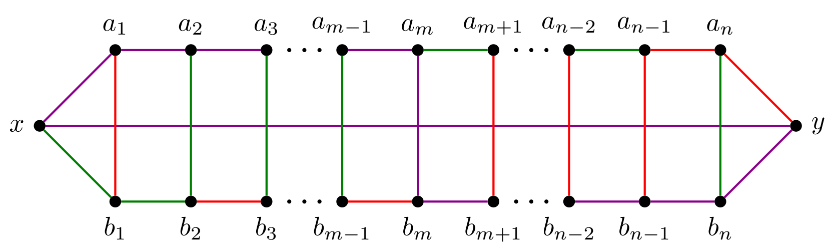
Proof: The cycle $C = xa_1a_2 \dots a_nyb_n \dots b_2b_1$ is a Hamiltonian cycle.



(a) A class of prism Hamiltonian decompositions of Figure 13 where n is odd. There is an even number of green vertical edges a_ib_i.



(b) A class of prism Hamiltonian decompositions of Figure 13 where n is even. There is an odd number of green vertical edges a_ib_i.



(c) Another class of prism Hamiltonian decompositions of Figure 13 where n is odd. The purple cycle stands for a B-Y cycle.

Fig. 14: Prism Hamiltonian decompositions of Figure 13.

When n is an odd number, we note that the cycle C has the odd diagonals x—y and a_i—b_i. We partition $\{1, 2, \dots, n\}$ into two sets U, V, where $|U|$ is even. The edges $E(C) \cup \{xy\} \cup \{a_ib_i \mid i \in U\}$ form a type G4 gadget (see B-Y-G edges in Figure 14(a)). The edges $E(C) \cup \{a_ib_i \mid i \in V\}$ form a type G3 gadget (see B-Y-R edges in Figure 14(a)). There are 2^{n-1} such prism Hamiltonian decompositions in total.

When n is an even number, all diagonals a_i—b_i will still be odd diagonals, but the "long diagonal" x—y will be an even diagonal. We partition $\{1, 2, \dots, n\}$ into two sets U, V, where $|U|$ and V are odd. The edges $E(C) \cup \{xy\} \cup \{a_ib_i \mid i \in U\}$ form a type G5 gadget (see B-Y-G edges in Figure 14(b)). The edges $E(C) \cup \{a_ib_i \mid i \in V\}$ form a type G3 gadget (see B-Y-R in Figure 14(b)). There are 2^{n-1} such prism Hamiltonian decompositions in total. $\square$

Remark 20 *Indeed, the graph in Figure 13 has many other decompositions. When n is odd, pick a number $m \in \{1, 2, \ldots, n\}$. The cycle $C' = xa_1a_2 \ldots a_mb_m \ldots b_{n-1}b_ny$ is an even cycle. The edges $E(C') \cup \{xb_1, b_1b_2\} \cup \{a_ib_i \mid 2 \leq i < m\} \cup \{a_ia_{i+1} \mid m \leq i \leq n-3\}$ form a type G2 gadget over a type G3 gadget (see B-Y-G edges in Figure 14(c)). The edges $E(C') \cup \{ya_n, a_na_{n-1}\} \cup \{a_ib_i \mid m < i \leq n-2\} \cup \{b_ib_{i+1} \mid 2 \leq i < m\}$ form a gadget of the same type (see B-Y-R edges in Figure 14(c)). When n is even, a similar construction exists.*

3 Concluding remarks

The variety of methods used to prove that certain cubic graphs are prism-decomposable probably shows that there is no single argument that will help us resolve the conjecture that all 3-connected cubic graphs are prism decomposable. It will probably end up in a situation similar to the famous graceful labelling problem of trees. That is many partial results proving that certain families of cubic graphs are Hamiltonian decomposable. For instance, does the existence of a Hamiltonian cycle in G help us prove that G is Hamiltonian decomposable? The Möbius Ladder casts doubt whether the Hamilton cycle can always be used.

We conclude by listing a sample of prism decomposable problems:

1. Cubic bipartite graphs.
2. Cubic planar graphs.
3. Cubic Hamiltonian graphs.
4. Cubic Hamiltonian planar graphs.
5. Cubic generalized Halin graphs of order $4k$.
6. Cubic generalized Halin graphs whose trees are caterpillars.
7. Decision problems:

D1: Given a cubic graph of order $2n$, is it a generalized Halin graph? That is does the graph contain a cycle C_{n+1} such that when its edges are deleted we are left with a tree?

This can be determined in linear time for Halin graphs.

D2: Given a cubic graph and a Hamiltonian cycle in it. Can the diagonals be partitioned into two sets such that each set plus the cycle is a spanning B-Y graph?

D3: Same as in D2 except the graph is planar. Note that in this case the diagonal splits into two non-intersecting sets.

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