

# Reachability of the phase space of the lit-only $\sigma$ -game on a graph

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## Abstract

## TODO: Abstract

*Keywords:* Edge clique partition, Line hypergraph, Sutner's Theorem, Transvection group, **TODO: Keywords**

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## 1. Introduction

Let  $V$  be a finite set. For each  $v \in V$ , we often write  $v$  for the set  $\{v\}$  and thus identify  $V$  with  $\binom{V}{1}$ . We view  $2^V$ , the power set of  $V$ , as a binary linear space in which the sum of any two elements of  $2^V$ , say  $x$  and  $y$ , denoted by  $x + y$ , is the symmetric difference of  $x$  and  $y$ . We use the symbol  $\equiv$  for equality/assignment in the binary field  $\mathbb{F}_2$ . For each  $v \in V$ , we define the binary linear functional  $v^*$  to be the map from  $2^V$  to  $\mathbb{F}_2$  such that

$$v^*(x) \equiv |\{v\} \cap x| \equiv \begin{cases} 1, & v \in x, \\ 0, & v \notin x, \end{cases}$$

for all  $x \in 2^V$ . In general, for each  $x \subseteq V$ , let  $x^* = \sum_{v \in x} v^*$ . It is convenient to write  $x^*(y) \in \mathbb{F}_2$  as  $\langle x, y \rangle$  for any  $x, y \in 2^V$ .

A *digraph*  $D$  is a pair of finite sets, its *vertex set*  $V_D$  and its *arc set*  $A_D \subseteq V_D \times V_D$ . We call  $u$  a *neighbour* of  $v \in V_D$  in the digraph  $D$  provided  $(v, u) \in A_D$ . The set of all neighbours of  $v$  in  $D$  is denoted  $N_D(v)$ . We define  $N_D \in \text{End}(2^{V_D})$  as the binary linear operator such that

$$N_D(x) = \sum_{v \in x} N_D(v) = \sum_{v \in V_D} v^*(x) N_D(v)$$

for each  $x \in 2^{V_D}$ . The *adjacency form* of the digraph  $D$ , denoted by  $\mathbb{A}_D$ , is the bilinear form on the binary space  $2^{V_D}$  such that

$$\mathbb{A}_D(x, y) \equiv |N_D(x) \cap y| \equiv y^*(N_D(x)) \equiv \langle y, N_D(x) \rangle$$

for all pairs  $x, y \in 2^{V_D}$ . If  $\mathbb{A}_D$  is a symmetric bilinear form, namely  $\mathbb{A}_D(x, y) \equiv \mathbb{A}_D(y, x)$  for all pairs  $x, y \in 2^{V_D}$ , or equivalently,  $\mathbb{A}_D(v, u) \equiv \mathbb{A}_D(u, v)$  for all  $\{u, v\} \in \binom{V_D}{2}$ , we say that  $D$  is a *symmetric digraph*.

Let  $D$  be a digraph. For every  $v \in V_D$ , construct a map  $\mathcal{T}_v \in \text{End}(2^{V_D})$  by setting

$$\mathcal{T}_v(x) = x + v^*(x) N_D(v)$$

for each  $x \in 2^{V_D}$ . The *phase space of the lit-only  $\sigma$ -game* on the digraph  $D$ , denoted by  $\mathcal{PS}_D$ , is the digraph with

- vertex set  $V_{\mathcal{PS}_D} = 2^{V_D}$ ,
- and arc set  $A_{\mathcal{PS}_D} = \{(x, \mathcal{T}_v(x)) \mid v \in x \in V_{\mathcal{PS}_D}\}$ .

The main object of this paper is reachability of the phase space. It is obvious that  $\mathcal{PS}_D$  is the Cartesian product [4] of  $\mathcal{PS}_E$  where  $E$  runs through all weakly connected components of  $D$ . And for a weakly connected digraph  $E$ ,  $\mathcal{PS}_E$  can be furthermore decomposed into  $\mathcal{PS}_F$ , where  $F$  runs through all strongly connected components of  $E$ , by a more complicated way. Therefore we just restrict our attention on the lit-only  $\sigma$ -game on a strongly connected digraph  $D$ . **we do not have a definition of connectedness yet.**

combination of two strands: the study of algebraic objects associated with graphs, the use of tools from algebra to derive properties of graphs. [3]

## 2. Notation

We undertake the necessary task of introducing some notation in this section. They are prepared not only for presenting our main results but also for facilitating our mathematical analysis.

**shall we define (vertex) induced digraph somewhere?**

### 2.1. Words

For each integer  $k$ , the set of first  $k$  positive integers is denoted  $[k]$ . Note that  $[0]$  is just the empty set  $\emptyset$ . Let  $V$  be a finite set. We view the set  $V^k$  of all words of length  $k$  over the alphabet  $V$  as  $V^{[k]}$  and so we often write  $W \in V^k$  as  $W_1 \cdots W_k$ . We record the set of all finite words over  $V$  as  $V^*$ , which is just  $\bigcup_{i=0}^{\infty} V^i$ .

### 2.2. Mixed graphs

A *mixed graph*  $M$  consists of a pair of finite sets, its *vertex* set  $V_M$  and its *face* set  $F_M$ , and two binary linear maps from  $2^{F_M}$  to  $2^{V_M}$ , its *head boundary map*  $\partial_M^+$  and *tail boundary map*  $\partial_M^-$ , which satisfy

$$|\partial_M^+(f)| = |\partial_M^-(f)|$$

and

$$|\partial_M^+(f) \cup \partial_M^-(f)| \in \{1, 2\}$$

for all  $f \in F_M$ . The *full boundary map*  $\partial_M$  of a mixed graph  $M$  is the binary linear map from  $2^{F_M}$  to  $2^{V_M}$  such that

$$\partial_M(f) = \partial_M^+(f) \cup \partial_M^-(f)$$

for all  $f \in F_M$ . We define the *head coboundary map*  $d_M^+$ , the *tail coboundary map*  $d_M^-$  and the *full coboundary map*  $d_M$  of a mixed graph  $M$  to be the adjoints of  $\partial_M^+$ ,  $\partial_M^-$  and  $\partial_M$ , respectively, as specified by

$$\langle d_M^\circ(v), e \rangle \equiv \langle v, \partial_M^\circ(e) \rangle \quad (2.1)$$

and so

$$d_M^\circ(v) = \sum_{v \in \partial_M^\circ(f), f \in F_M} f = \sum_{f \in F_M} v^*(\partial_M^\circ(f))f$$

for all  $v \in V_M, e \in F_M$  and  $(\partial^\circ, d^\circ) \in \{(\partial^+, d^+), (\partial^-, d^-), (\partial, d)\}$ . For each  $v \in V_M$ , we call  $d_M^+(v)$  and  $d_M^-(v)$  the set of *incoming faces* of  $M$  at  $v$  and the set of *outgoing faces* of  $M$  at  $v$ , respectively. The set of *out-neighbours* of a vertex  $v \in V_M$  in a mixed graph  $M$ , denoted by  $N_M^+(v)$ , is

$$\{w \in V_M \mid \exists f \in d_M^-(v) \cap d_M^+(w) \text{ s.t. } \partial_M(f) = \{v, w\}\};$$

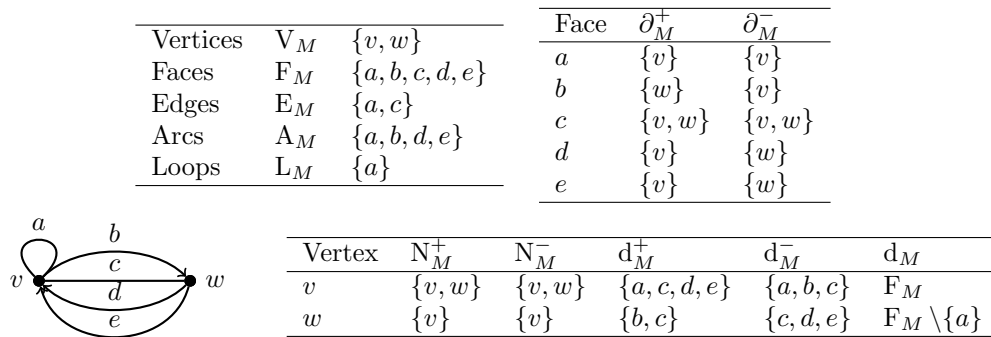
The set of *in-neighbours* of a vertex  $v \in V_M$  in a mixed graph  $M$ , denoted by  $N_M^-(v)$ , is

$$\{w \in V_M \mid \exists f \in d_M^+(v) \cap d_M^-(w) \text{ s.t. } \partial_M(f) = \{v, w\}\}.$$

Let  $N_M^+$  and  $N_M^-$  be two elements of  $\text{End}(2^{V_M})$  satisfying  $N_M^+(v) = N_M^+(v)$  and  $N_M^-(v) = N_M^-(v)$  for all  $v \in V_M$ . We often directly call an out-neighbour a neighbour and write  $N_M^+$  as  $N_M$ . A *walk of length  $k$*  in the mixed graph  $M$  is a word  $f_1 \dots f_k \in F_M^k$  such that  $\partial_M^+(f_i) \cap \partial_M^-(f_{i+1}) \neq \emptyset$  for  $i \in [k-1]$ . For a mixed graph  $M$ , its set of *edges*, denoted by  $E_M$ , is  $\{f \in F_M \mid \partial_M^+(f) = \partial_M^-(f)\}$ ; its set of *arcs*, denoted by  $A_M$ , is  $\{f \in F_M \mid |\partial_M^+(f)| = |\partial_M^-(f)| = 1\}$ ; and its set of *loops*, denoted by  $L_M$ , is  $E_M \cap A_M$ . Note that

$$F_M = E_M \cup A_M.$$

To illustrate the definition, we give an example in Fig. 2.1. We summarize the above terms and introduce some other terms on



**Figure 2.1:** A mixed graph  $M$  and its boundary maps.

a mixed graph  $M$  in Table 2.1. According to the distributions of its edges, arcs and loops, we call a mixed graph a *multidigraph*, a *multigraph*, a *simple mixed graph*, a *digraph* or a *graph* as described in Table 2.2.

**loopless graph, loop graph, pseudograph?**

Terminology	Notation	Definition / Condition
Vertex set	$V_M$	-
Face set	$F_M$	-
Edge set	$E_M$	$\{f \in F_M \mid \partial_M^+(f) = \partial_M^-(f)\}$
Arc set	$A_M$	$\{f \in F_M \mid  \partial_M^+(f)  =  \partial_M^-(f)  = 1\}$
Loop set	$L_M$	$E_M \cap A_M$
Proper edge set	-	$E_M \setminus A_M = E_M \setminus L_M$
Proper arc set	-	$A_M \setminus E_M = A_M \setminus L_M$
Head boundary map	$\partial_M^+$	-
Tail boundary map	$\partial_M^-$	-
Full boundary map	$\partial_M$	$\partial_M(x) = \sum_{f \in x} (\partial_M^+(f) \cup \partial_M^-(f))$
Head coboundary map	$d_M^+$	adjoint of $\partial_M^+$
Tail coboundary map	$d_M^-$	adjoint of $\partial_M^-$
Full coboundary map	$d_M$	adjoint of $\partial_M$
Walk	$f_1 \dots f_k$	$f_i \in F_M, \partial_M^+(f_i) \cap \partial_M^-(f_{i+1}) \neq \emptyset$

**Table 2.1:** Mixed graphs  $M$ .

Terminology	Notation	Definition / Condition
Mixed graph	$M, N, \dots$	-
Multidigraph	$D, E, \dots$	$F_M = A_M$ , or equivalently, $E_M = L_M$
Multigraph	$G, H, \dots$	$F_M = E_M$ , or equivalently, $A_M = L_M$
Simple mixed graph	$M, N, \dots$	$(\partial_M^-, \partial_M^+)$ is injective
Digraph	$D, E, \dots$	Simple multidigraph
Graph	$G, H, \dots$	Simple multigraph

**Table 2.2:** Mixed graphs and various subclasses of mixed graphs.

Loop-graphs and loop-multigraphs are also often known as pseudographs. <http://www.proofwiki.org/wiki/Definition:Loop-Graph>

For a sign  $\circ \in \{+, -\}$ , the  $\circ$ -incidence matrix (over  $\mathbb{F}_2$ ), denoted by  $\mathbb{B}_M^\circ$ , is an element of  $\mathbb{F}_2^{V_M \times F_M}$  that is the incidence matrix of the relation  $\partial^\circ$ . That is, the  $v$ -th row of  $\mathbb{B}_M^\circ$  can be said to be a coordinate representation of  $d_M^\circ(\chi_v)$  while the  $f$ -th column of  $\mathbb{B}_M^\circ$  is a coordinate representation of  $\partial_M^\circ(\chi_f)$ . It is clear that  $\mathbb{B}^+(M) = \mathbb{B}^-(M)$  if and only if  $M$  is a multigraph, i.e.,  $F_M = E_M$ .

The *adjacency matrix* (over  $\mathbb{F}_2$ ), denoted by  $\mathbb{A}_M$ , is the matrix whose  $v, w$ -entry is  $|\partial_M^-(v) \cap \partial_M^+(w) \cap d_M(\{v, w\})|$ .

The *Laplacian matrix* (over  $\mathbb{F}_2$ ), denoted by  $\Delta_M$ , is  $\mathbb{B}_M^-(\mathbb{B}_M^+)^T$ .

The *proper-edge-degree matrix* (over  $\mathbb{F}_2$ ), denoted by  $\mathbb{D}_M$  is a diagonal matrix whose  $v, v$ -entry is  $|\partial_M^-(v) \cap \partial_M^+(v) \setminus d_M(v)|$ .

These three matrices satisfy the identity

$$\Delta_M = \mathbb{D}_M - \mathbb{A}_M.$$

A mixed graph is *symmetric* if the adjacency matrix is a symmetric matrix.

An *isolated vertex* is a vertex  $v$  with  $N_M^-(v) \cup N_M^+(v) = \emptyset$ .

The *looped mixed graph*, denoted by  $\widehat{M}$ , is the graph with  $V_{\widehat{M}} = V_M$  and  $F_{\widehat{M}} = F_M \setminus L_M$ .

A mixed graph is *loop-forward* if every vertex can be reached by a loop vertex. A mixed graph is *loop-backward* if every vertex can be reach a loop vertex. A mixed graph is *loop-linked* if every vertex can reach and can be reached by a loop vertex.

**TODO: Loop-mixed graph is not good. A loop-forward loop-backward mixed graph may not be loop-linked. A mixed graph is loop-linked if and only if every strongly connected components has a loop**

In simple mixed graph, we treat  $L_M$  as subset of both  $V_M$  and  $F_M$ .

### 2.2.1. Graphs

**TODO**

For a graph  $G$ , we usually think of an edge  $e \in E_G$  as the set  $\partial_M^-(e) = \partial_M^+(e) \in \binom{V_G}{1} \cup \binom{V_G}{2}$  and in this way we can assume  $E_G \subseteq \binom{V_G}{1} \cup \binom{V_G}{2}$ .

### 2.2.2. Digraphs

**TODO**

For a digraph  $D$ , we always identify each arc  $e \in A_D$  with  $(\partial_D^-(e), \partial_D^+(e)) \in V_D \times V_D$  and hence naturally assume  $A_D \subseteq V_D \times V_D$ . We identify a symmetric digraph with the graph that shares the same adjacency matrix.

**The symmetrization of a simple mixed graph is the digraph ... A graph gives rise to a symmetric digraph..... The lit-only  $\sigma$ -game on a symmetric digraph .... We often do not distinguish A and B.**

Each mixed graph  $M$  is associated with a multidigraph  $\widetilde{M}$ , in which an edge of  $M$  is regarded as two arcs. **how to regard an edge as two arcs**

Let  $D$  be a digraph. For any  $S \subseteq V_D$ , let  $D \ominus S$  be the digraph obtained from  $D$  by deleting all those arcs out-going from  $S$ , namely

$$(V_{D \ominus S}, A_{D \ominus S}) = (V_D, A_D \cap ((V_D \setminus S) \times V_D)),$$

and let  $D - S$  be the digraph obtained from  $D$  by deleting  $S$  and all incident arcs, namely

$$(V_{D-S}, A_{D-S}) = (V_D \setminus S, A_D \cap ((V_D \setminus S) \times (V_D \setminus S))).$$

If there is a walk  $W = W_1 \cdots W_n$  in a digraph  $D$ , we say that  $\partial_D^-(W_1)$  can reach  $\partial_D^+(W_n)$  in  $D$  via  $W$ . For any  $v \in V_D$ , we say that  $v$  can reach itself via the walk of length 0, namely the empty word in  $A_D^0$ . If  $u$  and  $v$  are two vertices in  $D$  such that each can reach the other in  $D$ , we say that  $u$  and  $v$  are reachable to each other in  $D$  and express it by  $u \rightsquigarrow v$ . The reachability relation in a digraph  $D$  is clearly an equivalence relation and each equivalence class is called a *strongly connected component* of  $D$ . Each equivalence class of the reachability relation in the symmetrization digraph of  $D$  is called a *weakly connected component* of  $D$ .

Assume that  $v$  is a loop vertex of a digraph  $D$ . Let  $D \boxminus v$  be the digraph with vertex set  $V_D \setminus \{v\}$  and arc set  $A_D + N_D^-(v) \times N_D^+(v)$ . It is clear that, for a graph  $G$  with a loop vertex  $v$ ,  $E_{G \boxminus v} = E_G + \binom{N_G^-(v)}{1} + \binom{N_G^+(v)}{2}$ .

### 2.3. Hypergraphs and line graphs

**what about just defining line digraphs of a mixed graph and then regard the symmetric digraph as graph (in case that the mixed graph is symmetric)? or should we only consider digraphs? do we need to call the line digraphs as line graphs? is there a structural characterization of the line digraphs of mixed graphs?**

The *line digraph* of a mixed graph  $M$ , denoted by  $\mathfrak{L}(M)$ , is the digraph with vertex set  $V(\mathfrak{L}(M)) = F_M$  and arc set

$$A_{\mathfrak{L}(M)} = \{(a, b) \in F_M \times F_M \mid |\partial_M^+(a) \cap \partial_M^-(b)| = 1\}.$$

It is clear that

$$N_{\mathfrak{L}(M)} = d_M^- \partial_M^+ \quad (2.2)$$

and that

$$A_{\mathfrak{L}(M)}(x, y) \equiv \langle y, N_{\mathfrak{L}(M)}(x) \rangle \equiv \langle y, d_M^- \partial_M^+(x) \rangle \equiv \langle \partial_M^-(y), \partial_M^+(x) \rangle \equiv \langle \partial_M^+(x), \partial_M^-(y) \rangle \quad (2.3)$$

for all  $x, y \in F_M$ .

“Line graphs” are just the graphs corresponding to the “line multigraphs” of multigraphs. There are two ways to treat multigraphs as graphs, hence they lead to two definitions of “line graphs”. One is doing the calculation over the Boolean semifield, and the other uses the binary field. The former one gives the widely used definition of line graphs, which is good at handling “combinatorial properties”, and we use the latter one to give our own definition of line graphs, which prefers the “algebraic properties”. These two definitions coincide for graphs, but they deal with parallel edges differently.

A *hypergraph*  $H$  is a pair of finite sets, the *vertex set*  $V_H$  and the *hyperedge set*  $E_H$ , and a *boundary map*  $\partial_H : E_H \rightarrow 2^{V_H} \setminus \{\emptyset\}$ . A *simple hypergraph* is a hypergraph whose boundary map is injective. A *k-hypergraph* is a hypergraph  $H$  with  $\text{Im } \partial_H \subseteq \binom{V_H}{\leq k}$ . The *k-line hypergraph* of a hypergraph  $H$ , denoted by  $\mathfrak{L}_k(H)$ , is the simple *k-hypergraph* with

- $V_{\mathfrak{L}_k(H)} = E_H$ ,
- $E_{\mathfrak{L}_k(H)} = \{S \subseteq E_H \mid |\bigcap_{e \in S} \partial_H(e)| \equiv 1\}$

and  $\partial_{\mathfrak{L}_k(H)}$  the inclusion map.

The *line graph* of a multigraph  $G$  (namely 2-hypergraph), denoted by  $\mathfrak{L}(G)$ , is the 2-line hypergraph of the multigraph. A *line graph* is the line graph of some multigraph. An *ordinary line graph* is the line graph of some graph.

The adjacency matrix of the line graph of a multigraph  $G$  is

$$A_{\mathfrak{L}(G)} = \mathbb{B}_G^\top \mathbb{B}_G.$$

**Remark 2.1.** For a mixed graph  $M$ ,  $\mathfrak{L}(M)$  is a **TO BE DEFINED: digraph minor** of  $\mathfrak{L}(\widetilde{M})$ . In particular, for a multigraph  $M$ ,  $\widetilde{\mathfrak{L}(M)}$  is a digraph minor of  $\mathfrak{L}(\widetilde{M})$ , namely, this definition is compatible with line graphs.

Since the characteristic of  $\mathbb{F}_2$  is two, our definition of line graphs leads to

$$A(\mathfrak{L}(M)) = \mathbb{B}^+(M)^\top \mathbb{B}^-(M), \quad (2.4)$$

and

$$\Delta(M) = \mathbb{B}^-(M) \mathbb{B}^+(M)^\top. \quad (2.5)$$

## 2.4. Lit-only $\sigma$ -game

It is a transvection when  $v \notin L_D$ , and a projection when  $v \in L_D$ .

## 3. Main results

### 3.1. Line graphs

**Theorem 3.1** (Generalization of Krausz's characterization). *A graph is an ordinary line graph if and only if it has an edge clique partition.*

A minor extension of Whitney's theorem [11].

**Theorem 3.2** (Generalization of Whitney's theorem). *Except four graphs, an ordinary line graph has a unique edge clique partition.*

**Theorem 3.3** (Generalization of Beineke's characterization). *A graph is a line graph if and only if it does not contain any graph in Figs. 4.1 and 4.2 as a vertex induced subgraph.*

### 3.2. Critical subgraphs

#### sub-mixedgraph, subgraph or subdigraph???

A *non-singular digraph* is a digraph whose adjacency matrix has full rank over  $\mathbb{F}_2$ . A *critical subdigraph* of a digraph  $E$  is a nonsingular induced subdigraph  $D$  of  $E$  fulfilling  $\text{rank } \mathbb{A}(D) = \text{rank } \mathbb{A}(E)$ . **And a *critical subgraph* of a graph is a critical subdigraph of it.** – **It is not quite correct. We will talk about symmetric critical subdigraph. Indeed, we may not really need the concept of a graph.** A class  $\mathcal{C}$  of digraphs is *critical closed* if every digraph in  $\mathcal{C}$  has a critical subdigraph which also falls in  $\mathcal{C}$ .

**Theorem 3.4.** *The following classes of digraphs are critical closed.*

- (Connected) loopless line graphs.
- (Connected) line graphs with loops.
- (Connected) loopless non-line graphs.
- (Connected) non-line graphs with loops.

Critical subgraphs play an important role in this paper. Due to it, together with characterization of line graphs, we are able to divide the line graph case and the non-line graph case, and then handle them separately. For Theorem ?? we only have one proof which makes use of the existence of critical subgraphs. For both Theorem ?? and Theorem ??, we will supply two proofs, one of which depends on the existence of critical subgraphs and the other is by induction on the number of vertices; the former walks along a shorter route while the latter approach might be applied in the digraph case.

The connected line graph  $G$  of a multigraph  $H$  is an *emerald line graph* if  $H$  has odd number of vertices.

**Theorem 3.5.** *Let  $M$  be a strongly connected mixed graph and let  $N = \mathfrak{L}(M)$ . Take  $S \subseteq V_M$ .*

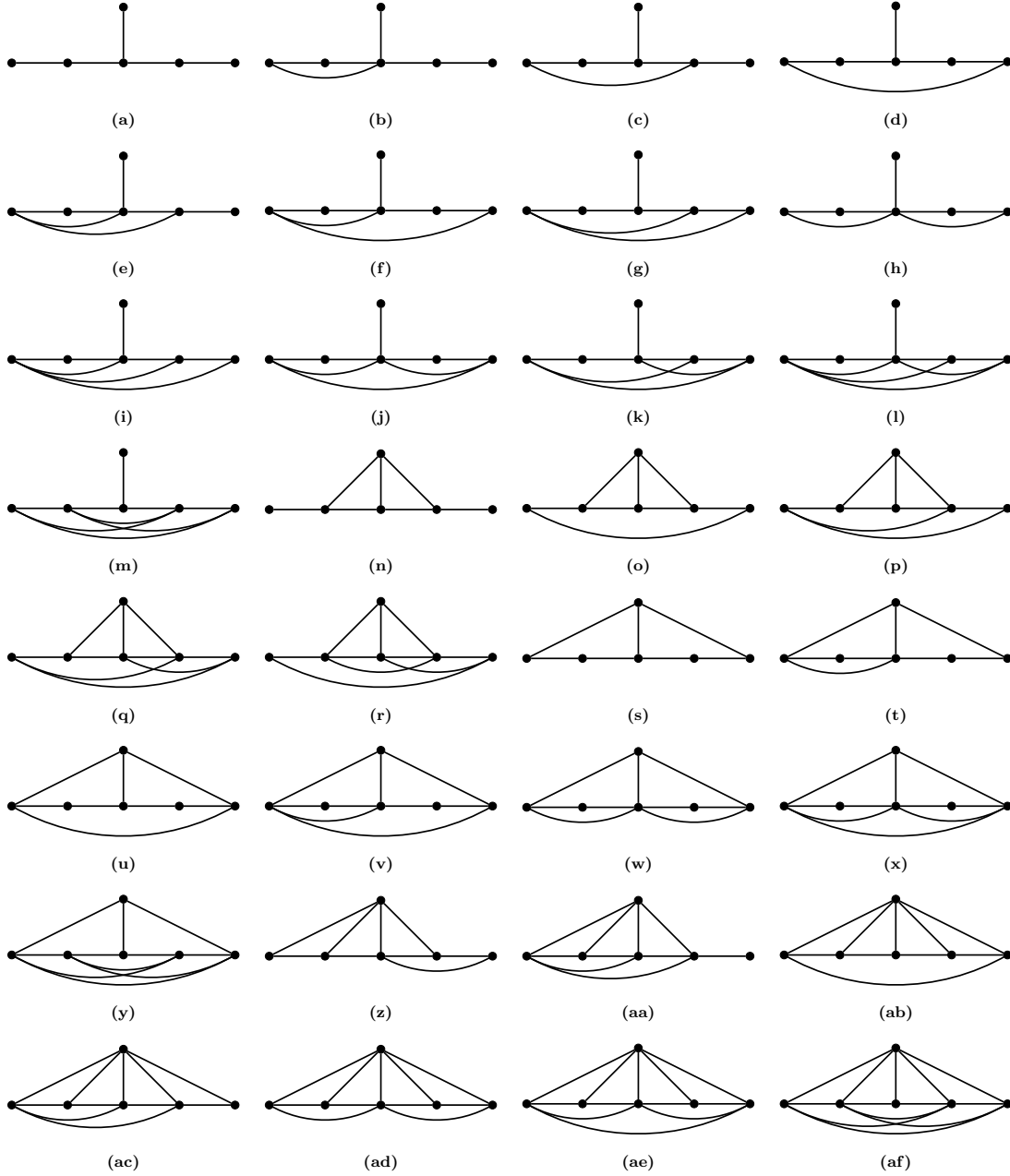
- *If  $A_M \neq \emptyset$ , then  $N[S]$  is a strongly connected critical subgraph of  $N$  if and only if  $M[S]$  is a strongly connected mixed graph with vertex set  $V_M$  and  $S \cap A_M \neq \emptyset$ .*
- *If  $A_M = \emptyset$  and  $|V_M|$  is odd, then  $N[S]$  is a strongly connected critical subgraph of  $N$  if and only if  $M[S]$  is a strongly connected graph with vertex set  $V_M$ .*
- *If  $A_M = \emptyset$  and  $|V_M|$  is even, then  $N[S]$  is a strongly connected critical subgraph of  $N$  if and only if  $M[S]$  is a strongly connected graph and  $S \cap d_M(v) \neq \emptyset$  for all  $v \in V_M$  with at most one exception.*

## 4. Characterizations of line graphs

The ultimate purpose of this section is to prove Theorem 3.3. We split the Theorem 3.3 into two parts, Theorems 4.1 and 4.2.

**Theorem 4.1.** *A loopless graph is a line graph if and only if it does not contain any graph in Fig. 4.1 as a vertex induced subgraph.*

**Theorem 4.2.** *A loop-linked graph is a line graph if and only if it does not contain any graph in Fig. 4.2 as a vertex induced subgraph.*



**Figure 4.1:** The 32 minimal forbidden vertex induced subgraphs of loopless line graphs.

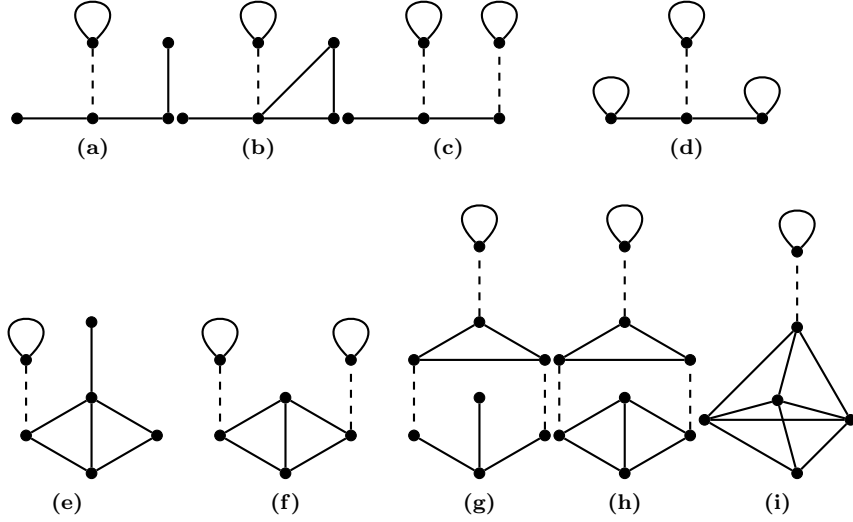
**Remark 4.3.** In this paper, we use the dashed line in a figure to mean a path of length zero or more. If the figure contains a dashed line, this figure should be understood as a class of infinitely many graphs, namely those graphs in which the dashed lines are replaced by paths of all possible lengths.

**Remark 4.4.** We display in Fig. 4.2 infinitely many forbidden subgraphs as nine classes. We remark that a graph in one class may have a graph in another class as a vertex induced subgraph. It is routine work to further single out those minimal forbidden vertex induced subgraphs for loop-linked line graphs. We do not pursue it as it will cost much more space to list all those minimal forbidden vertex induced subgraphs and the list in Fig. 4.2 fits well for our purpose in our current work on the lit-only  $\sigma$ -game.

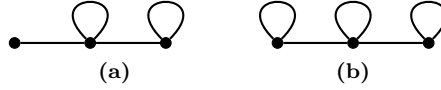
**TODO:** Take the definition of  $\boxminus$  out of the section.

We define another operator  $\boxminus$  in §4.12. Similar with operator  $-$ , which removes a specified vertex from the graph,  $\boxminus$  removes a specified loop vertex from the graph in a different way. With the help of it, we give another characterization of non-line graphs.

**Theorem 4.5.** A connected loop-linked graphs is not a line graph if and only if it can be reduced to either Fig. 4.3(a) or Fig. 4.3(b) by a sequence of operators  $-$  and  $\boxminus$  through connected loop-linked graphs.



**Figure 4.2:** The 9 classes of forbidden vertex induced subgraphs of loop-linked graphs. Dashed lines stand for paths of length zero or more.



**Figure 4.3:** The two endings.

During the process of proving Theorem 3.3, we develop some concepts and characterizations. Here is the outline of the section. The terminologies in the quotation marks are defined later.

§4.1: A graph is line graph if and only if the “reduced graph” “associated to it” is a line graph.

§4.2: “Reduced line graphs” and ordinary line graphs are basically the same.

§4.3: Beineke’s characterization.

§4.4: Proof of Theorem 4.1. Some corollaries and problems.

§4.5: The line graph operator is the composition of two operators, “\*” and “ $\mathfrak{T}$ ”.

§4.6: “Root graphs” are graphs whose line graphs are isomorphic to a given ordinary line graph.

§4.7: “Edge clique partitions” are hypergraphs whose image under “ $\mathfrak{T}$ ” is a given ordinary line graph.

§4.8: “Root graphs” and “edge clique partitions” are image of each other under operator “\*”. Proof of Theorem 3.1.

§4.9: Uniqueness of “edge clique partitions”. Proof of Theorem 3.2.

§4.10: Linear time algorithm to find all “edge clique partitions”.

§4.11: Proof of Theorem 4.2.

§4.12: Properties of operator “ $\square$ ”. Proof of Theorem 4.5.

#### 4.1. Reduced graphs

For a graph  $G$  and  $u, v \in V_G$ , denote the fact of  $N_G(u) = N_G(v)$  by  $u \sim_G v$ , or simply  $u \sim v$ . It is clear that  $\sim$  is an equivalence relation. Denote the  $\sim$ -equivalence class containing vertex  $v$  by  $\bar{v}$ . Define the *quotient graph (over  $\sim$ )*,  $G/\sim$ , by setting  $V_{G/\sim} = \{\bar{v} \mid v \in V_G\}$  and  $E_{G/\sim} = \{\{\bar{u}, \bar{v}\} \mid \{u, v\} \in E_G\}$ . Two distinct vertices  $u$  and  $v$  of a graph  $G$  are *twin vertices* in  $G$  if  $u \sim_G v$ . A graph  $G$  is *reduced* if  $G \cong G/\sim$ , namely you cannot find any twin in  $G$ .

**Lemma 4.6.** *Let  $G$  be a multigraph. For any  $S \subseteq E_G$ , we have  $\mathfrak{L}(G)[S] = \mathfrak{L}(G[S])$ .*

*Proof.* We just need to do a transparently simple verification by definition. □

**Lemma 4.7.** *A graph  $G$  is a line graph if and only if the quotient graph  $G/\sim$  is a line graph.*

*Proof.* Let  $S$  be a transversal of the set of  $\sim$ -equivalence classes of  $G$ . We can check that  $G/\sim$  is isomorphic to  $G[S]$ . Consequently, if  $G$  is a line graph, we will know from Lemma 4.6 that  $G/\sim$  is also a line graph.

To prove the reverse direction, we suppose that  $G/\sim$  is the line graph of a multigraph  $H$  at this time. Construct a multigraph  $H'$  by setting  $V_{H'} = V_H$ ,  $E_{H'} = V_G$  and  $\partial_{H'}(v) = \partial_H(\bar{v})$  for each  $v \in V_G$ . It is immediate that  $G = \mathfrak{L}(H')$ . □



#### 4.2. Ordinary line graphs

Recall that, a line graph is *ordinary* if it is the line graph of a graph

**Lemma 4.8.** *Line graphs and ordinary line graphs are closed under taking vertex induced subgraphs.*

*Proof.* Easy corollary of Lemma 4.6. □

**Lemma 4.9.** *Except the path  $P_3$ , the cycle  $C_4$ , the diamond (Fig. 4.11(b)), the wheel  $W_4$  (Fig. 4.11(d)) and the line graph of  $K_4$ , which are ordinary but not reduced, a line graph is ordinary if and only if it is reduced.*

*Proof.* Parallel edges would introduce twins in the line graph, hence, reduced line graphs are always reduced.

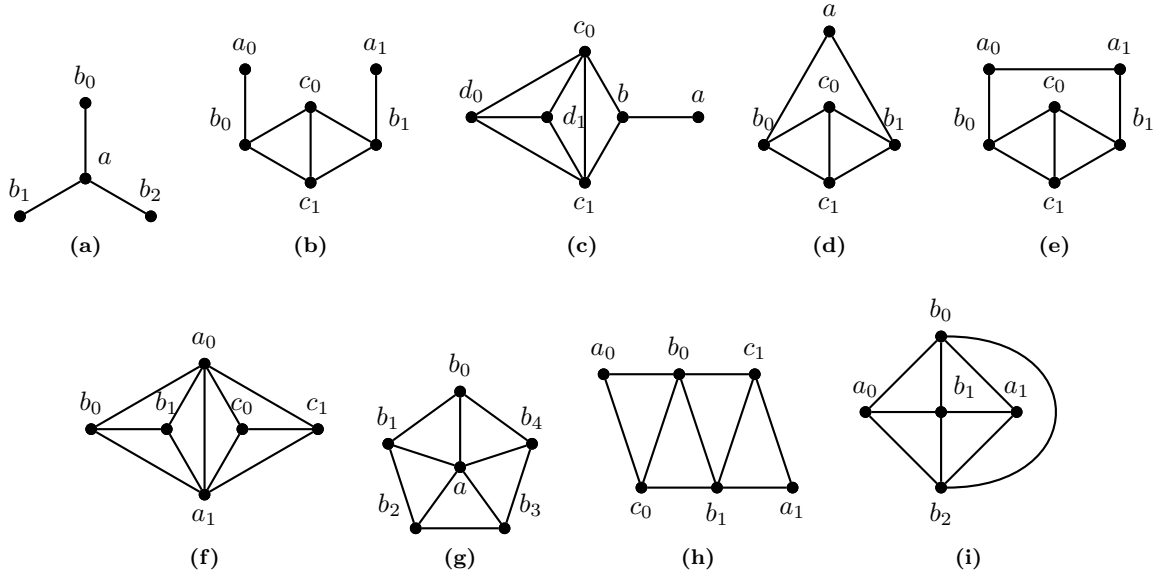
Suppose two edges  $e, f$  in a graph  $G$  form a twin in the line graph  $\mathfrak{L}(G)$ . Then,  $V_G = \partial_G(e) \cup \partial_G(f)$ , hence  $|V_G| \leq 4$ . □

**Remark 4.10.** *The five exceptions in Lemma 4.9 are line graphs of connected loopless graphs having spanning edge induced  $2P_2$ .*

#### 4.3. Forbidden vertex induced subgraphs of loopless ordinary line graphs

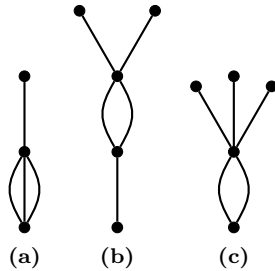
Beineke's forbidden-subgraph characterization of line graphs [1, 2]. For another characterization, see [7, 9].

**Theorem 4.11.** [10, Theorem 7.1.18] *A loopless graph is an ordinary line graph if and only if it does not contain any of the nine graphs in Fig. 4.4 as a vertex induced subgraph.*



**Figure 4.4:** Nine forbidden vertex induced subgraphs in Beineke's line graph characterization. The labelling is for the proof of Lemma 4.34 and Lemma 4.35.

**Example 4.12.** *Figs. 4.4(a), 4.4(d) and 4.4(i) are line graphs of Figs. 4.5(a), 4.5(b) and 4.5(c), respectively.*



**Figure 4.5:** Root multigraphs of three forbidden vertex induced subgraphs of ordinary line graphs.

#### 4.4. Forbidden vertex induced subgraphs of loopless line graphs

**Lemma 4.13.** *Let  $G$  be a reduced loopless graph. The following statements are equivalent.*

- $G$  contains one graph in Fig. 4.4 as a vertex induced subgraph.
- $G$  contains one graph in Fig. 4.1 as a vertex induced subgraph.

*Proof.* First, we explain the strategy of the proof. Suppose  $H$  is a non-reduced loopless graph. Pick a twin  $v, w$  from  $H$ . Let  $S_H$  be the set of all possible supgraphs of  $H$  with vertex set  $V_H \cup \{u\}$  and in which  $v, w$  do not form a twin. Clearly,  $|S| \leq 2^{|V_H| - 1}$ . Thus, every reduced graph containing  $H$  as vertex induced subgraph must contain one of  $S$  as vertex induced subgraph. For reduced loopless graph  $H$ , we set  $S_H = \{H\}$ .

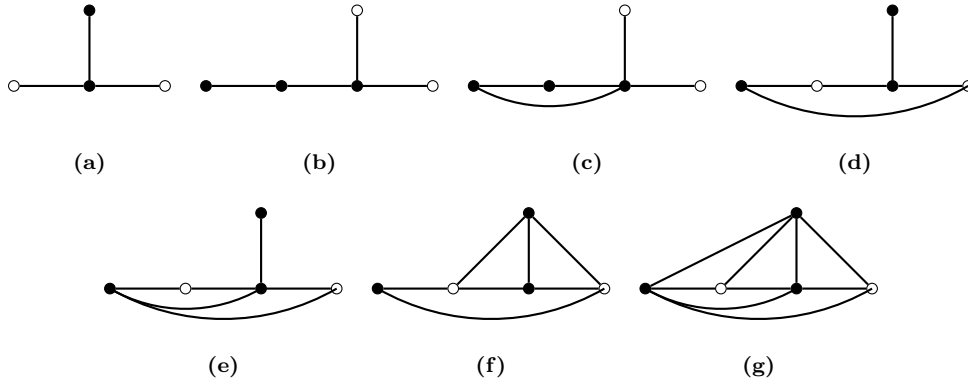
Fig. 4.6 shows the  $H$  we used, and Table 4.1 shows the corresponding  $S_H$ .

**TODO: Check the table**

Smaller graph	Bigger graph
Fig. 4.4(a) = Fig. 4.6(a)	Figs. 4.6(b), 4.6(c), 4.6(d), 4.6(e)
Fig. 4.6(b)	Figs. 4.1(a), 4.1(b), 4.1(c), 4.1(d), 4.1(e), 4.1(f), 4.1(g), 4.1(h)
Fig. 4.6(c)	Figs. 4.1(b), 4.1(f), 4.1(h), 4.1(j), 4.1(k), 4.1(l)
Fig. 4.6(d)	Figs. 4.1(c), 4.1(f), 4.1(g), 4.1(k), 4.1(s), 4.1(t), 4.1(u), 4.1(v)
Fig. 4.6(e)	Figs. 4.1(e), 4.1(i), 4.1(j), 4.1(l), 4.1(t), 4.1(v), 4.1(w), 4.1(x)
Fig. 4.4(b)	Fig. 4.1(n)
Fig. 4.4(c)	Fig. 4.1(m)
Fig. 4.4(d) = Fig. 4.6(f)	Figs. 4.1(k), 4.1(p), 4.1(q), 4.1(r), 4.1(v), 4.1(y)
Fig. 4.4(e)	Fig. 4.1(o)
Fig. 4.4(f)	Fig. 4.1(ad)
Fig. 4.4(g)	Fig. 4.1(ab)
Fig. 4.4(h)	Fig. 4.1(z)
Fig. 4.4(i) = Fig. 4.6(g)	Figs. 4.1(aa), 4.1(ac), 4.1(ae), 4.1(af)

**Table 4.1:** No reduced graph can have a graph in the first column as a vertex induced subgraph but does not have the corresponding graph in the second column as a vertex induced subgraph.

□



**Figure 4.6:** A pair of circles stands for a twin.

**Theorem 4.14.** *Let  $G$  be a loopless graph. The following statements are equivalent.*

- (1) The graph  $G$  is a line graph.
- (2) The graph  $G / \sim$  is a line graph.
- (3) The graph  $G / \sim$  is an ordinary line graph.
- (4) The graph  $G / \sim$  does not contain any graph in Fig. 4.4.
- (5) The graph  $G / \sim$  does not contain any graph in Fig. 4.1.
- (6) The graph  $G$  does not contain any graph in Fig. 4.1 as a vertex induced subgraph.

*Proof.* (1)  $\Leftrightarrow$  (2): Lemma 4.7.

(2)  $\Leftrightarrow$  (3): Lemma 4.9.

(3)  $\Leftrightarrow$  (4): Theorem 4.11.

(4)  $\Leftrightarrow$  (5): Lemma 4.13.

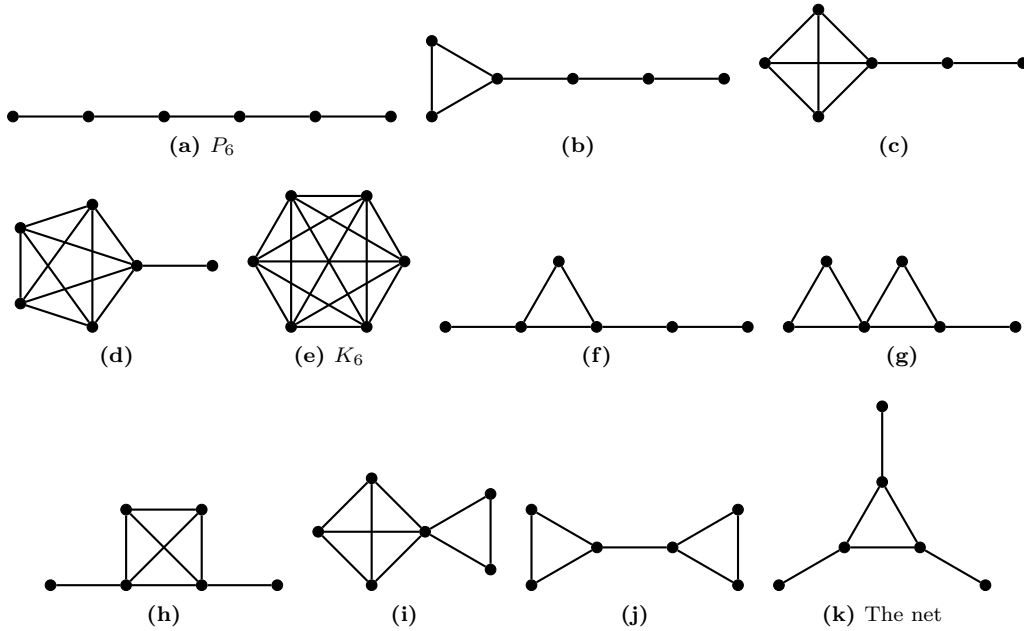
(5)  $\Leftrightarrow$  (6): All graphs in Fig. 4.1 are reduced graphs. □

*Proof of Theorem 4.1.* The equivalence between the statements (1) and (7) in Theorem 4.14. □

**Corollary 4.15.** *For a loopless graph  $G$ , the following statements are equivalent.*

- *The graph  $G$  is a line graph.*
- *The graph  $G$  does not contain any graph in Fig. 4.1 as a vertex induced subgraph.*
- *Every 6-vertex vertex induced subgraph of  $G$  is a line graph.*
- *Every 6-vertex connected vertex induced subgraph of  $G$  is a line graph.*
- *Every 6-vertex connected nonsingular vertex induced subgraph of  $G$  is the line graph of a 7-vertex tree.*
- *Every 6-vertex connected nonsingular vertex induced subgraph of  $G$  is listed in Fig. 4.7.*

**Corollary 4.16.** *Every loopless graph  $G$  with  $\text{rank } \mathbb{A}_G \leq 5$  is a line graph. In particular, each loopless graph with at most 5 vertices is a line graph.*



**Figure 4.7:** Eleven 6-vertex line graphs of trees.

**Problem 4.17.** *Why are minimal forbidden subgraphs of loopless line graphs nonsingular? Why do all minimal forbidden subgraphs of loopless line graphs have 6 vertices? Why does the set of 6-vertex nonsingular loopless graphs coincide with the disjoint union of the set of 6-vertex line graphs of trees (Fig. 4.7) and the set of all minimal forbidden subgraphs of loopless line graphs (Fig. 4.1)? Is this related to the root system of type  $E_6$ ?*

#### 4.5. Line hypergraphs

Let  $H$  be a hypergraph. The *coboundary map*, denoted by  $d_H$ , maps a vertex  $v$  to  $\{e \in E_H \mid v \in \partial_H(e)\}$ . Here, we define four operators the *dual*  $\cdot^*$ ,  $\mathfrak{T}'$ ,  $\bar{\cdot}$ ,  $\cdot|_k$  and  $\mathfrak{T}$  as follow.

- $V_{H^*} = E_H$ ,  $E_{H^*} = V_H$  and  $\partial_{H^*} = d_H$ .
- $V_{\mathfrak{T}'(H)} = V_H$ ,  $E_{\mathfrak{T}'(H)} = \{(f, e) \mid \emptyset \neq f \subseteq \partial_H(e), e \in E_H\}$  and  $\partial_{\mathfrak{T}'(H)}(f, e) = f$ .
- $V_{\bar{H}} = V_H$ ,  $E_{\bar{H}} = \{e \subseteq V_H \mid |\partial_H^{-1}(e)| = 1 \pmod{2}\}$  and  $\partial_{\bar{H}}$  the inclusion map.

- $V_{H|_k} = V_H$ ,  $E_{H|_k} = \partial_H^{-1} \left( \binom{V_H}{\leq k} \right)$  and  $\partial_{H|_k}$  the restriction.
- $\mathfrak{T} = (\cdot|_2) \circ (\cdot) \circ \mathfrak{T}'$ .

**Lemma 4.18.**  $\mathfrak{L}_k = (\cdot|_k) \circ (\cdot) \circ \mathfrak{T}' \circ (\cdot^*)$ . In particular,  $\mathfrak{L} = \mathfrak{T} \circ (\cdot^*)$ .

**Theorem 4.19.** Every  $k$ -hypergraph is the  $k$ -line hypergraph of some  $j$ -hypergraph.

Two different vertices form a *co-twin* (**TODO: not good**), if their coboundaries are the same. The dual of a hypergraph is simple if and only if the hypergraph has no co-twins. Co-twin-free multigraphs are multigraphs without “isolated proper edges”.

**TODO: The definition of  $\cdot^*$ .**

Since we only consider simple co-twin-free hypergraphs in the rest of the section, we identify a simple hypergraph  $H$  with its hyperedge set  $E_H \subseteq 2^{V_H}$ , and let  $E_{H^*} = \{d_H(v) \mid v \in V_H\}$ . We still have  $\mathfrak{L} = \mathfrak{T} \circ (\cdot^*)$ . For each hypergraph isomorphism  $f$  from  $G$  to  $H$ , we can naturally lift it to a hypergraph isomorphism  $f^*$  from  $G^*$  to  $H^*$ .

#### 4.6. Root graphs

Recall that, an *isolated vertex* in a multigraph  $G$  is a vertex  $v \in V(G)$  with  $N_G(v) = \emptyset$ . A *root multigraph* of a line graph  $G$  without isolated vertices, is a multigraph  $R$ , without isolated vertices, equipped with a graph isomorphism  $\ell_R$  from  $\mathfrak{L}(R)$  to  $G$ . We may thus view  $R$  as a multigraph whose edges are labeled by  $V_G$  according to  $\ell_R$ . A *root graph* of is a root multigraph that is a graph. A *root graph isomorphism* from  $S$  to  $T$  is a graph isomorphism  $f$  from  $S$  to  $T$  such that  $\ell_S = \ell_T \circ f^*$ .

From the definition, we can see that  $G = (\ell_R \circ \mathfrak{T} \circ *) (R)$ . Define  $\mathfrak{R}_G(R) = \ell_R(R^*)$ , hence we have  $\mathfrak{T}(\mathfrak{R}_G(R)) = G$ .

**Lemma 4.20.** The operator  $\mathfrak{R}_G$  is injective up to the isomorphism of root graphs.

*Proof.* If for two root graphs  $S, T$ ,  $\mathfrak{R}_G(S) = \mathfrak{R}_G(T)$ , then  $d_T \circ \ell_T^{-1} \circ \ell_S \circ d_S^{-1}$  gives a root graph isomorphism from  $S$  to  $T$ .  $\square$

The concept of root graphs could be defined for line graphs with isolated vertices, however, it is almost meaningless and the construction of operator  $\mathfrak{J}, \mathfrak{R}$  and the definition of edge clique partition would become much more complicated.

#### 4.7. Edge clique partitions

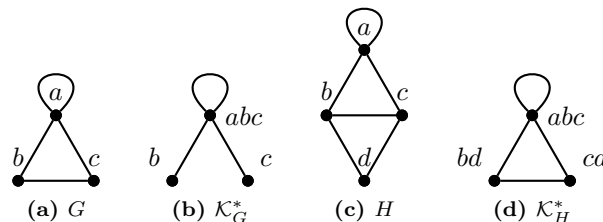
Krausz [5] discovers a characterization of loopless ordinary line graphs in terms of edge clique partition (See [6, Theorem 1.13] or [10, Theorem 7.1.16]), which is later used by Roussopoulos [8] to design a linear time algorithm for recognizing loopless ordinary line graphs and reconstructing their root graphs. This and next subsection will discuss parallel results for general ordinary line graphs (possibly having loops).

Assuming that  $G$  is an ordinary line graph and  $R$  is a root graph of  $G$ , to capture the essence of this construction of  $\mathfrak{R}(R)$ , we make the following definition which is a variant of the usual concept for edge clique partitions of loopless graphs originally introduced by Krausz [5] and will play key role in the derivation of a Krausz-type characterization [6, p. 15] of line graphs in our setting.

**Definition 4.21.** Let  $G$  be a graph without isolated vertices. An edge clique partition of  $G$  is a simple hypergraph  $K$  on the vertex set  $V_G$  and hyperedge set  $\mathcal{K} \subseteq 2^{V_G} \setminus \{\emptyset\}$  such that the following conditions hold:

- For every non-loop vertex  $v$ , it holds  $d_K(v) \in \binom{\mathcal{K}}{2}$ ;
- For every loop vertex  $v$ , it holds  $d_K(v) \in \binom{\mathcal{K}}{1}$ ;
- The edge set  $E_G$  is the disjoint union of  $(k \cap L_G) \cup \binom{k}{2}$  where  $k$  runs through all elements of  $\mathcal{K}$ ;
- For each  $k \in \mathcal{K}$ , it happens  $|k \cap L_G| \leq 1$ .

**Example 4.22.** In Fig. 4.8 we demonstrate two graphs with loops,  $G$  and  $H$ . They both have a unique edge clique partition (see Theorem 4.32), denoted by  $\mathcal{K}_G$  and  $\mathcal{K}_H$ , respectively. The dual hypergraphs of them are also depicted in Fig. 4.8. Observe that  $\mathfrak{T}(\mathcal{K}_G) = G$ ,  $\mathfrak{T}(\mathcal{K}_H) = H$ ,  $\mathfrak{L}(\mathcal{K}_G^*) \cong G$  and  $\mathfrak{L}(\mathcal{K}_H^*) \cong H$ .



**Figure 4.8:** Graphs, edge clique partitions and root graphs.

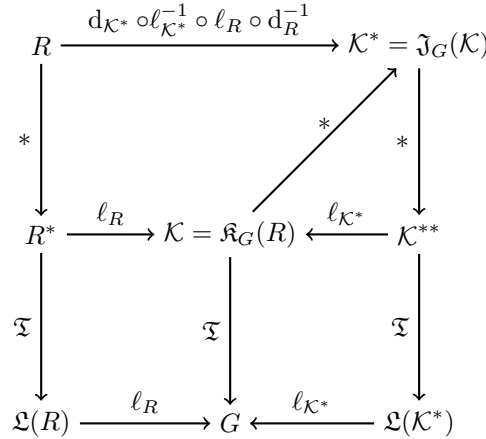
**Remark 4.23.** Given any root graph  $R$  of a graph  $G$  without isolated vertices, we can check that  $\mathfrak{R}(R)$  is an edge clique partition of  $G$ .

**Remark 4.24.** Suppose that  $\mathfrak{K}$  is an edge clique partition of a graph  $G$ . Here are some quick consequences of Definition 4.21.

- (1) Condition (ii) together with Condition (iii) says that  $d_{\mathcal{K}}(v) = \{N_G(v)\}$  for every loop vertex  $v$ .
- (2) For two different non-loop vertices  $v, w$ , Condition (iii) implies that they cannot appear together in two elements of  $\mathcal{K}$  and so it follows from Condition (i) that  $d_{\mathcal{K}}(v) \neq d_{\mathcal{K}}(w)$ ; For two loop vertices  $v, w$ , Condition (iv) guarantees that  $d_{\mathcal{K}}(v) \neq d_{\mathcal{K}}(w)$ .
- (3) The dual hypergraph  $\mathcal{K}^*$  of  $\mathcal{K}$  is indeed a graph, as Conditions (i) and (ii) demonstrate that  $E_{\mathcal{K}^*} = \{d_{\mathcal{K}}(v) \mid v \in V_G = V_{\mathcal{K}}\} \subseteq \binom{V_G}{1} \cup \binom{V_G}{2}$ .
- (4) Combining the above three claims gives the fact that  $d_{\mathcal{K}}$  is an injective map from  $V_G$  to  $\binom{V_G}{1} \cup \binom{V_G}{2}$ .
- (5) For non-loop vertex  $v$ , Conditions (i) and (iii) show that  $\bigcup_{k \in d_{\mathcal{K}}(v)} k = N_G(v) \cup \{v\}$  and  $\bigcap_{k \in d_{\mathcal{K}}(v)} k = \{v\}$ .
- (6) The Gaifman graph of a hypergraph  $H$  is the graph with  $V_H$  as vertex set and with an edge between two distinct vertices if they appear together in the same hyperedge of  $H$ . It follows from Condition (iii) that the Gaifman graph of  $\mathcal{K}$  is the lopped graph  $\hat{G}$ .

**Remark 4.25.**  $\mathfrak{T}(\mathcal{K}) = G$ ,  $\mathfrak{L}(\mathcal{K}^*) = \mathfrak{T}(\mathcal{K}^{**}) \cong \mathfrak{T}(\mathcal{K}) = G$ .

#### 4.8. The bridge between root graphs and edge clique partitions



**Figure 4.9:**  $G$  is a line graph without isolated vertices.  $R$  is a root graph of  $G$ .  $\mathcal{K}$  is an edge clique partition of  $G$ . An arrow  $a \xrightarrow{f} b$  stands for  $f(a) = b$ .

At this point, let us introduce one more operator to better understand Example 4.22. Given an edge clique partition  $\mathcal{K}$  of a graph  $G$  without isolated vertices, in view of Remark 4.24 (3)(4), we can define a graph  $\mathfrak{J}_G(\mathcal{K}) = \mathcal{K}^*$  equipped with a mapping  $\ell_{\mathfrak{J}(\mathcal{K})} = d_{\mathcal{K}}^{-1}$ . One sees that it happens  $\mathcal{K}_G = \mathfrak{R}_G(\mathfrak{J}_G(\mathcal{K}_G))$  and  $\mathcal{K}_H = \mathfrak{R}_H(\mathfrak{J}_H(\mathcal{K}_H))$ .

**Remark 4.26.** Given any edge clique partition  $\mathcal{K}$  of a graph  $G$  without isolated vertices, we can check that  $\mathcal{K} = \mathfrak{R}_G(\mathfrak{J}_G(\mathcal{K}))$ .

With only a little more effort, we can make use of  $\mathfrak{J}$  and  $\mathfrak{R}$  to derive the following result, which provides an understanding of Example 4.22 and bridges the concepts of root graphs and edge clique partitions.

**Theorem 4.27.** Let  $G$  be a graph without isolated vertices. The operators  $\mathfrak{J}_G$  and  $\mathfrak{R}_G$  are inverse of each other up to the isomorphism of the root graphs. Hence,  $\mathfrak{R}_G$  is a bijection from edge clique partitions to root graphs.

*Proof.* • Given a root graph  $R$ ,  $\mathfrak{R}_G(R)$  is an edge clique partition. Remark 4.23.

- Given an edge clique partition  $\mathcal{K}$ ,  $\mathfrak{J}_G(\mathcal{K})$  is a root graph. Remarks 4.26, 4.25.
- Given a root graph  $R$ ,  $\mathfrak{J}_G(\mathfrak{R}_G(R)) \cong R$ . Lemma 4.20 and Remark 4.26.
- Given an edge clique partition  $\mathcal{K}$ ,  $\mathfrak{R}_G(\mathfrak{J}_G(\mathcal{K})) = \mathcal{K}$ . Remark 4.26.

□

*Proof of Theorem 3.1.* Corollary of the Theorem 4.27.

□

#### 4.9. The uniqueness of edge clique partition

**Lemma 4.28.** *Let  $G$  be a connected graph and let  $S$  be a subset of  $V_G$ . There is at most one edge clique partition of  $G$  containing  $S$  as a hyperedge.*

*Proof.* Let us prove by contradiction. Suppose that  $\mathcal{K}$  and  $\mathcal{K}'$  are two different edge clique partitions of  $G$  containing  $S$ . This implies that  $S \neq \emptyset$  and that there exists  $v \in V_G$  so that  $d_{\mathcal{K}}(v)$  and  $d_{\mathcal{K}'}(v)$  are different subsets of  $2^{V_G}$ . Take  $w \in S$ . Since the graph  $G$  is connected, we can thus find a path  $w = v_0, v_1, \dots, v_t = v$ . By Remark 4.24 (1) and (5), whenever we know that  $k \in \mathcal{K} \cap \mathcal{K}'$ , we can conclude that  $d_{\mathcal{K}}(w) = d_{\mathcal{K}'}(w)$  for all  $w \in k$ . From  $S \in \mathcal{K} \cap \mathcal{K}'$  we are now able to derive  $d_{\mathcal{K}}(v_0) = d_{\mathcal{K}'}(v_0)$ . Then, we can pick  $k \in \mathcal{K}$  such that  $v_1 \in k \in d_{\mathcal{K}}(v_0) = d_{\mathcal{K}'}(v_0)$ , hence  $d_{\mathcal{K}}(v_1) = d_{\mathcal{K}'}(v_1)$ . Performing an inductive reasoning along this line yields finally  $d_{\mathcal{K}}(v_t) = d_{\mathcal{K}'}(v_t)$ , contradicting with our assumption that  $d_{\mathcal{K}}(v) \neq d_{\mathcal{K}'}(v)$ .  $\square$

A *clique* of a graph is a subset of its vertices such that every two vertices in the subset form an edge of the graph.

**Lemma 4.29.** *Suppose that  $G$  is a graph possessing an edge clique partition  $\mathcal{K}$ . If  $S$  is a maximal clique of  $G$  of size not 3, then it happens  $S \in \mathcal{K}$ .*

*Proof.* The result is trivial when  $|S| \leq 1$ . Assume that  $|S| = 2$  or  $|S| \geq 4$ .

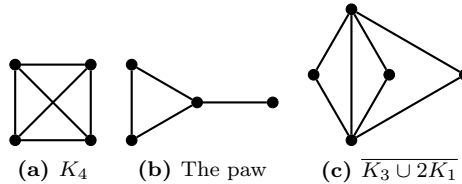
**Case 1:**  $S \cap L_G \neq \emptyset$

Pick a loop vertex  $v$  from the clique  $S$ . Remark 4.24(1) tells us that,  $N_G(v) \in \mathcal{K}$  and it is the unique maximal clique containing  $v$ . Therefore,  $S = N_G(v) \in \mathcal{K}$ .

**Case 2:**  $S \cap L_G = \emptyset$

We aim to reach a contradiction under the assumption of  $S \notin \mathcal{K}$ .

Pick a vertex  $v$  from the clique  $S$ . Because  $S$  is a maximal clique and  $S \notin \mathcal{K}$ , we know from Remark 4.24 (5) that  $d_{\mathcal{K}}(v)$  contains two elements  $A, B$ , such that  $A' = A \cap S \setminus \{v\}$  and  $B' = B \cap S \setminus \{v\}$  gives a partition of  $S \setminus \{v\}$  into two nonempty parts. This cannot happen when  $|S| = 2$ . Without loss of generality, as  $|S| \geq 4$ , assume that we can take  $\{a_1, a_2\} \in \binom{A'}{2}$  and  $b \in B'$ . By now, since  $a_1$  and  $a_2$  appear together in  $A \in \mathcal{K}$ , Condition (iii) shows that, there are two different hyperedge  $B_1, B_2 \in \mathcal{K}$  such that  $\{b, a_1\} \subseteq B_1$ ,  $\{b, a_2\} \subseteq B_2$ , hence  $B, B_1, B_2$  are three different elements in  $d_{\mathcal{K}}(b)$ . This contradicts with the fact that  $d_{\mathcal{K}}(b) \in \binom{\mathcal{K}}{\leq 2}$ .  $\square$



**Figure 4.10:** Three forbidden vertex induced subgraphs for triangle-free graphs.

**Lemma 4.30.** *Any graph that contains one of Fig. 4.10 as induced subgraph has at most one edge clique partition.*

*Proof.* **Case 1:** The graph has an induced Fig. 4.10(a).

Find a maximal clique containing the induced Fig. 4.10(a). The result follows from Lemma 4.28 and 4.29.

**Case 2:** The graph has an induced Fig. 4.10(b).

It is easy to see that every edge clique partition has an element containing the  $K_3$  in Fig. 4.10(b). The result follows from Lemma 4.28 or 4.29.

**Case 3:** The graph has an induced Fig. 4.10(c).

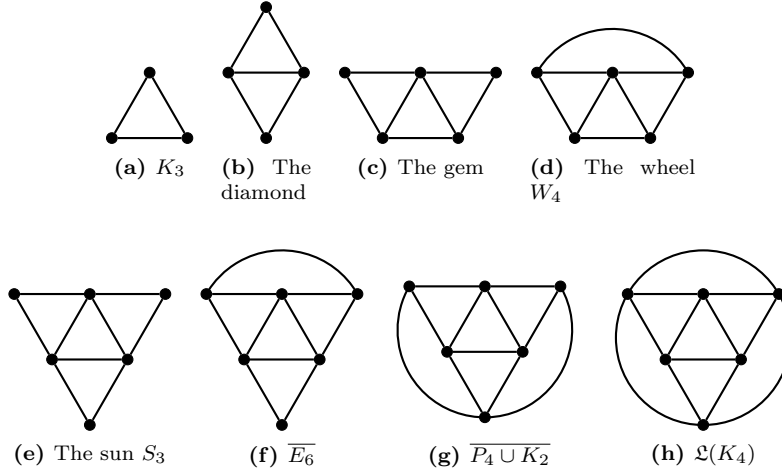
Fig. 4.10c has no edge clique partition. Therefore, neither does the graph.  $\square$

**Theorem 4.31.**  $\{\text{Connected loopless (Fig. 4.10)-free graphs}\} = \{\text{connected loopless triangle-free graphs}\} \cup \text{Fig. 4.11}.$

*Proof.* Suppose graph  $G$  is a connected loopless non-triangle-free (Fig. 4.10)-free graph, and  $S = \{a, b, c\}$  be an induced  $K_3$ . Since Fig. 4.10(a) and 4.10(b) do not appear, the set  $(N_G(a) \cup N_G(b) \cup N_G(c)) \setminus S$  is the disjoint union of  $X = N_G(b) \cap N_G(c)$ ,  $Y = N_G(a) \cap N_G(c)$  and  $Z = N_G(a) \cap N_G(b)$ . The non-existence of Fig. 4.10(a) and 4.10(c) makes  $|X|, |Y|, |Z| \leq 1$ , and then the non-existence of Fig. 4.10(b) ensures  $V_G = S \cup X \cup Y \cup Z$ . Finally, we can check the graphs in Fig. 4.11 are the only possibilities.  $\square$

**TODO:** Name a hypergraph whose hyperedge set gives a cover of its vertex set.

**Theorem 4.32.** *Let  $G$  be a connected ordinary line graph without isolated vertices. Then  $G$  either has a unique edge clique partition or has exactly two edge clique partitions and is isomorphic to one graph in the first column of Table 4.2.*



**Figure 4.11:** Difference between connected loopless (Fig. 4.10)-free graphs and connected loopless triangle-free graphs.

	Graph	Type A ECP	Type B ECP
$X = \emptyset$ $Y = \emptyset$ $Z = \emptyset$			
$X = \{x\}$ $Y = \emptyset$ $Z = \emptyset$			
$X = \emptyset$ $Y = \{y\}$ $Z = \{z\}$			
$X = \{x\}$ $Y = \{y\}$ $Z = \{z\}$			

**Table 4.2:** Graphs and the dual hypergraphs of their two edge clique partitions (ECP).

*Proof.* According to Theorem 4.27,  $G$  has an edge clique partition. We note that all the graphs shown in the first column of Table 4.2 is loopless and has clique number three. It is also easy to check that each graph shown in the first column of Table 4.2 has exactly two edge clique partitions as indicated in the second and the third column the table.

**Case 1:**  $L_G \neq \emptyset$ .

Pick  $v \in L_G$ . By Remark 4.24 (1), each edge clique partition of  $G$  contains  $N_G(v)$  as a hyperedge. The uniqueness of the edge clique partition now follows from Lemma 4.28.

**Case 2:**  $L_G = \emptyset$ .

If the graph  $G$  is triangle-free, the result follows Lemma 4.29. If one of Fig. 4.10 appears as an induced subgraph, the result follows from Lemma 4.30. Due to 4.31, Fig. 4.11 shows all non-triangle-free (Fig. 4.10)-free graphs. It is easy to check that each of Figs. 4.11(c), 4.11(e), 4.11(f) and 4.11(g) has a unique edge clique partition.  $\square$

#### 4.10. Linear time construction of edge clique partition

Based on lemmas and theorems in §4.9, it is easy to develop a very simple linear time algorithm to find all edge clique partitions, and hence root graphs, of a given graph. The correctness is guaranteed by the previous proofs.

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**Algorithm 4.1** The unique edge clique partition containing a given hyperedge

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**Input:** Connected graph without isolated vertices  $G$ , a non-empty set  $S$  of vertices of  $G$

**Output:** The unique edge clique partition  $\mathcal{K}$  that contains  $S$ , or no edge clique partitions

```

1: function ECP-HELPER( $G, S$ )
2:   if  $S$  does not induce a clique, or  $S$  contains two loop vertices then
3:     return No edge clique partitions
4:    $\mathcal{K} \leftarrow \{S\}$ 
5:   while Exists a non-loop vertex  $v$  which is covered exactly once by elements in  $\mathcal{K}$  do
6:      $P \leftarrow$  the element containing  $v$ 
7:      $Q \leftarrow (N_G(v) \setminus P) \cup \{v\}$ 
8:     if  $Q$  does not induce a clique, or  $Q$  contains two loop vertices then
9:       return No edge clique partitions
10:     $\mathcal{K} \leftarrow \mathcal{K} \cup \{Q\}$ 
11:    if A loop vertex is covered twice, or a non-loop vertex is covered twice then
12:      return No edge clique partitions
13:  return  $\mathcal{K}$ 

```

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**Algorithm 4.2** All edge clique partitions

---

**Input:** Connected graph without isolated vertices  $G$

**Output:** All edge clique partitions

```

1: function ECP( $G$ )
2:    $S \leftarrow$  a maximal clique
3:   if  $|S| \neq 3$  then
4:     return ECP-HELPER( $G, S$ )
5:   else
6:     Suppose  $S = \{a, b, c\}$ 
7:      $T \leftarrow (N_G(b) \cap N_G(c)) \cup \{b, c\}$ 
8:     return ECP-HELPER( $G, S$ ) and ECP-HELPER( $G, T$ )

```

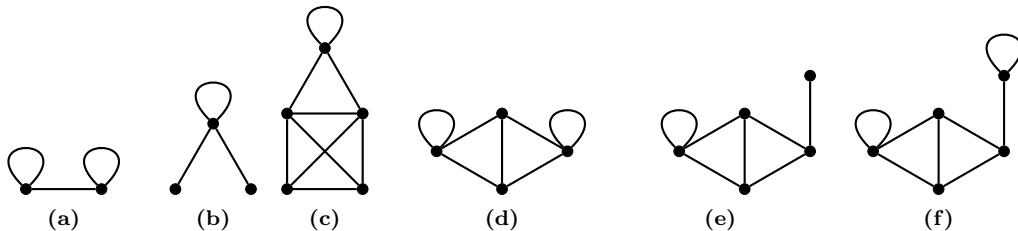
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#### 4.11. Forbidden vertex induced subgraphs of loop-linked line graphs

Recall that, the *lopped graph* of a graph  $G$ , denoted by  $\widehat{G}$ , is the loopless graph obtained by removing all loops from  $G$ .

**Lemma 4.33.** *If  $G$  is a line graph, then so is  $\widehat{G}$ . If  $G$  is an ordinary line graph, then so is  $\widehat{G}$ .*

*Proof.* We assume that  $G = \mathcal{L}(H)$  for a multigraph  $H$ . Take a set  $L$  which is disjoint with  $V_H$  and we insist that there is a bijection  $\gamma$  from  $L_G$  to  $L$ . Let  $H'$  be the graph with  $V_{H'} = V_H \cup L$ ,  $E_{H'} = E_H$ ,  $\partial_{H'}(e) = \partial_H(e)$  for  $e \in E_H \setminus L_G$  and  $\partial_{H'}(l) = \{l, \gamma(l)\}$  for  $l \in L_G$ . It is no hard to check that  $\widehat{G} = \mathcal{L}(H')$ , proving that  $H$  is a line graph. □

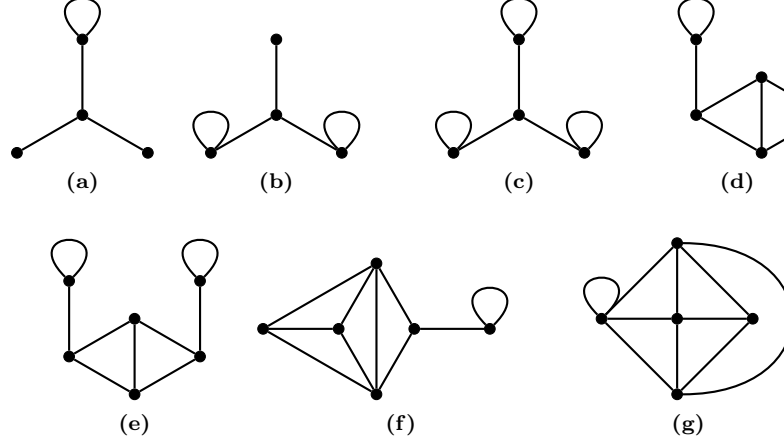


**Figure 4.12:** Six forbidden vertex induced subgraphs used in Lemma 4.34.

**Lemma 4.34.** *Let  $G$  be a graph without isolated vertices. The following statements are equivalent.*



- (1) The graph  $G$  has an edge clique partition.
- (2) The lopped graph  $\widehat{G}$  is an ordinary line graph and the graph  $G$  does not contain any graph in Fig. 4.12 as a vertex induced subgraph.
- (3) The lopped graph  $\widehat{G}$  does not contain any graph in Fig. 4.4 as a vertex induced subgraph and the graph  $G$  does not contain any graph in Fig. 4.12 as a vertex induced subgraph.
- (4) The graph  $G$  does not contain any graph displayed in Figs. 4.4, 4.12 or 4.13 as a vertex induced subgraph.



**Figure 4.13:** Seven forbidden vertex induced subgraphs used in Lemmas 4.34 and 4.36.

*Proof.* The result is trivial when  $G$  is a single loop. And we only need to consider connected graphs. Assume from now on that  $G$  is a connected graph and has at least two vertices, and  $\widehat{G}$  does not contain isolated vertices.

(1)  $\Rightarrow$  (2): Firstly, Theorem 4.27 together with Lemma 4.33 shows that  $\widehat{G}$  is an ordinary line graph.

Secondly, we can check that no graph in Fig. 4.12 is an ordinary line graph; Indeed, Fig. 4.12 (c), (d), (e), (f) are even not line graphs. Remember that, by Theorem 4.27,  $G$  is an ordinary line graph. Therefore, by Lemma 4.8, no graph in Fig. 4.12 can appear as a vertex induced subgraph of  $G$ .

(2)  $\Rightarrow$  (1): By Theorem 4.27,  $\widehat{G}$  has an edge clique partition  $\mathcal{K}$ .

**Case 1:**  $\mathcal{K} \supseteq \binom{L_G}{1}$ .

We assert that  $\mathcal{K} \setminus \binom{L_G}{1}$  gives an edge clique partition of  $G$ . We need to verify the four conditions in Definition 4.21. The first three conditions follow directly while Condition (iv) is ensured by the fact that  $G$  does not contain Fig. 4.12(a) as a vertex induced subgraph.

**Case 2:**  $\mathcal{K} \not\supseteq \binom{L_G}{1}$ .

We claim that the graph  $G$  is either Fig. 4.8(a) or Fig. 4.8(c). This will prove (1) as we already show in Example 4.22 that both Fig. 4.8(a) and Fig. 4.8(c) have an edge clique partition.

The assumption of  $\mathcal{K} \not\supseteq \binom{L_G}{1}$  means that there exists  $v \in L_G$  such that  $\{v\} \notin \mathcal{K}$ . Let  $\{U, V\} = d_{\mathcal{K}}(v) \in \binom{\mathcal{K}}{2}$ . It is clear that  $|U|, |V| \geq 2$ . Fig. 4.12(b) is a forbidden subgraph for  $G$ , and so  $U \cup V$  forms a maximal clique in  $G$  of size  $|U| + |V| - 1$ . If  $\max(|U|, |V|) \geq 3$ , then Lemma 4.28 applies to say that  $U \cup V \in \mathcal{K}$ , contradicting with  $\{U, V\} = d_{\mathcal{K}}(v)$ . Consequently, we arrive at  $|U| = |V| = 2$ .

We further observe that no vertex in  $U \triangle V$  can fall in  $L_G$ , as Fig. 4.12(a) cannot appear as a vertex induced subgraph of  $G$ . Let  $W \in \mathcal{K}$  be the clique covering the two non-loop vertices in  $U \cup V$ . As Fig. 4.12(d) is a forbidden subgraph of  $G$ , we know that  $W \cap L_G = \emptyset$ . Finally, due to the assumption that Fig. 4.12(c) is not an induced subgraph of  $G$ , it must happen  $|W| \leq 3$ .

**Case 2.1:**  $|W| = 2$ .

By the connectedness of  $G$ , the graph  $G$  is Fig. 4.8(a).

**Case 2.2:**  $|W| = 3$ .

Let  $\{u\} = W \setminus (U \cup V)$  and let  $W'$  be the clique such that  $\{W, W'\} = d_{\mathcal{K}}(u)$ . If  $|W'| > 1$ , then either Fig. 4.12(e) or Fig. 4.12(f) appears as a vertex induced subgraph of  $G$ , violating our assumption. But if  $|W'| = 1$ , then the connectedness of  $G$  forces  $G$  to be Fig. 4.8(c).

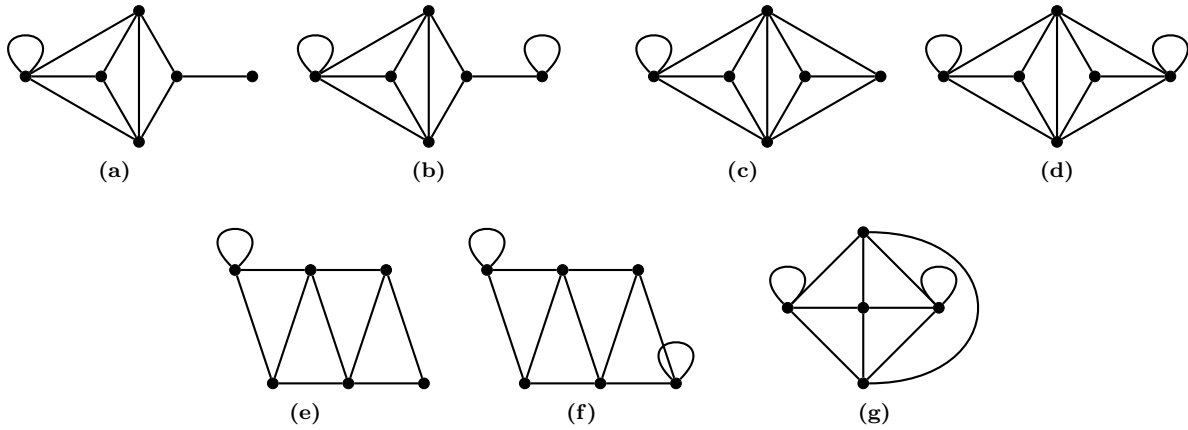
(2)  $\Leftrightarrow$  (3): This follows from Theorem 4.11.

(3)  $\Rightarrow$  (4): Note that if  $K$  is a vertex induced subgraph of  $G$  then  $\widehat{K}$  is a vertex induced subgraph of  $\widehat{G}$ . Table 4.3 shows that, when  $K$  is a graph in Fig. 4.13,  $\widehat{K}$  is a graph in Fig. 4.4, which completes the proof.

$\widehat{G}$	$L_G$	$G$	$G$ contains
Fig. 4.4(a)	$\{b_0\}$	Fig. 4.13(a)	
Fig. 4.4(a)	$\{b_1, b_2\}$	Fig. 4.13(b)	
Fig. 4.4(a)	$\{b_0, b_1, b_2\}$	Fig. 4.13(c)	
Fig. 4.4(b)	$\{a_0\}$	Fig. 4.13(d)	
Fig. 4.4(b)	$\{a_0, a_1\}$	Fig. 4.13(e)	
Fig. 4.4(c)	$\{a\}$	Fig. 4.13(f)	
Fig. 4.4(c)	$\{d_0\}$	Fig. 4.14(a)	Fig. 4.12(e)
Fig. 4.4(c)	$\{a, d_0\}$	Fig. 4.14(b)	Fig. 4.12(f)
Fig. 4.4(f)	$\{b_0\}$	Fig. 4.14(c)	Fig. 4.12(c)
Fig. 4.4(f)	$\{b_0, c_0\}$	Fig. 4.14(d)	Fig. 4.12(d)
Fig. 4.4(h)	$\{a_0\}$	Fig. 4.14(e)	Fig. 4.12(e)
Fig. 4.4(h)	$\{a_0, a_1\}$	Fig. 4.14(f)	Fig. 4.12(f)
Fig. 4.4(i)	$\{a_0\}$	Fig. 4.13(g)	
Fig. 4.4(i)	$\{a_0, a_1\}$	Fig. 4.14(g)	Fig. 4.12(d)

**Table 4.3:** The relation among Figs. 4.4, 4.12, 4.13 and 4.14.

(4)  $\Rightarrow$  (3): Our task is to demonstrate that  $G$  contains a graph in Figs. 4.12 and 4.13 as a vertex induced graph, provided  $\widehat{G}$  is a graph in Fig. 4.4 and  $L_G \neq \emptyset$ .



**Figure 4.14:** Some graphs arising from graphs in Fig. 4.4 by adding loops.

Let us assume that  $G$  contains neither 4.12(a) nor 4.12(b) as a vertex induced subgraph, namely no two different loops of  $G$  can be adjacent and the neighborhood of a loop in  $G$  must be a clique. We list all possibilities for the graphs  $\widehat{G}$  and  $G$  in Table 4.3. We are now done by the observation: every graph in Fig. 4.14 contains a graph in Fig. 4.12 as a vertex induced subgraph.  $\square$

**Lemma 4.35.** *Every loop-linked graph which contains a graph in Fig. 4.4 as a vertex induced subgraph must contain a graph in Fig. 4.15(a), 4.15(b), 4.15(c) or 4.15(d) as a vertex induced subgraph.*

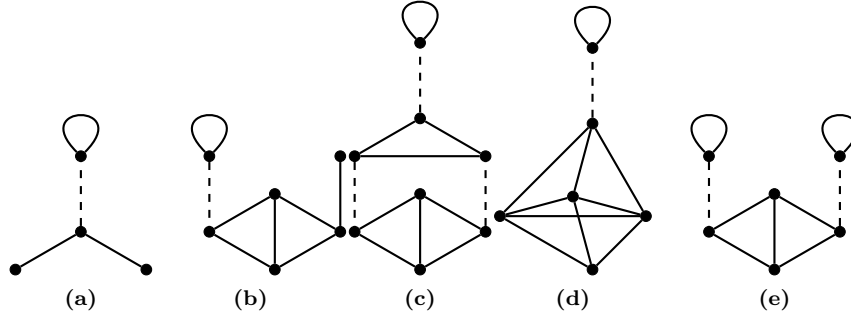
*Proof.* Let  $G$  be a graph having a vertex induced subgraph  $H$  which coincides with one graph in Fig. 4.4. Suppose  $G$  does not contain any graph in Fig. 4.15(a) as a vertex induced subgraph.

Loop-linkedness of  $G$  enable us to find a shortest path  $P$  which connects a vertex in  $L_G$  and a vertex in  $\cup_{u \in V_H} N_G(u)$ . Let  $v$  be an endpoint of  $P$  that has neighbors in  $V_H$ . Let  $N = N_G(v) \cap V_H$ . Since no graph represented by Fig. 4.15(a) can appear as a vertex induced subgraph of  $G$ ,  $N$  must induce a non-empty clique of  $H$ , and for every  $w \in N$ ,  $N_H(w) \setminus N$  induces a clique of  $H$  as well.

A case by case checking will show us that, there exists a set  $S \subseteq V_H$ , such that the vertex induced subgraph  $G[V_P \cup S]$  is in Fig. 4.15, arriving at the desired result. Table 4.4 provides the details for this case by case checking in which we enumerate all possible  $H$  and  $N$  and we also describe the asserted vertex induced subgraph we find.  $\square$

**Lemma 4.36.** *Let  $G$  be a loop-linked graph. Then the following two conditions are equivalent.*

- (1) *The graph  $G$  does not contain any graph displayed in Figs. 4.4, 4.12 or 4.13 as a vertex induced subgraph.*



**Figure 4.15:** Five classes used in Lemmas 4.35, 4.36 and 4.37. Dashed lines in the figure stand for paths of length zero or more.

$H$	$G[N]$	$N$	$S \subseteq V_H$	$G[V_P \cup S]$ is in
Fig. 4.4(a)	$K_1$	$\{b_0\}$	$\{a, b_0, b_1, b_2\}$	Fig. 4.15(a)
Fig. 4.4(b)	$K_1$	$\{a_0\}$	$\{a_0, a_1, b_0, b_1, c_0, c_1\}$	Fig. 4.15(b)
Fig. 4.4(b)	$K_2$	$\{a_0, b_0\}$	$\{a_1, b_0, b_1, c_0, c_1\}$	Fig. 4.15(b)
Fig. 4.4(b)	$K_3$	$\{b_0, c_0, c_1\}$	$\{a_1, b_1, c_0, c_1\}$	Fig. 4.15(b)
Fig. 4.4(c)	$K_1$	$\{a\}$	$\{a, b, c_0, c_1, d_0, d_1\}$	Fig. 4.15(c)
Fig. 4.4(c)	$K_1$	$\{d_0\}$	$\{a, b, c_0, c_1, d_0\}$	Fig. 4.15(b)
Fig. 4.4(c)	$K_2$	$\{a, b\}$	$\{b, c_0, c_1, d_0\}$	Fig. 4.15(c)
Fig. 4.4(c)	$K_2$	$\{d_0, d_1\}$	$\{a, b, c_0, c_1, d_0\}$	Fig. 4.15(b)
Fig. 4.4(c)	$K_3$	$\{b, c_0, c_1\}$	$\{c_0, c_1, d_0, d_1\}$	Fig. 4.15(c)
Fig. 4.4(c)	$K_3$	$\{c_0, d_0, d_1\}$	$\{c_0, c_1, d_0, d_1\}$	Fig. 4.15(d)
Fig. 4.4(c)	$K_4$	$\{c_0, c_1, d_0, d_1\}$	$\{a, b, c_0, c_1\}$	Fig. 4.15(b)
Fig. 4.4(d)	$K_2$	$\{a, b_0\}$	$\{a, b_0, b_1, c_0, c_1\}$	Fig. 4.15(c)
Fig. 4.4(d)	$K_3$	$\{b_0, c_0, c_1\}$	$\{a, b_1, c_0, c_1\}$	Fig. 4.15(b)
Fig. 4.4(e)	$K_2$	$\{a_0, a_1\}$	$\{a_0, a_1, b_0, b_1, c_0, c_1\}$	Fig. 4.15(c)
Fig. 4.4(e)	$K_2$	$\{a_0, b_0\}$	$\{a_1, b_0, b_1, c_0, c_1\}$	Fig. 4.15(b)
Fig. 4.4(e)	$K_3$	$\{b_0, c_0, c_1\}$	$\{a_1, b_1, c_0, c_1\}$	Fig. 4.15(b)
Fig. 4.4(f)	$K_1$	$\{b_0\}$	$\{a_0, a_1, b_0, c_0, c_1\}$	Fig. 4.15(c)
Fig. 4.4(f)	$K_2$	$\{b_0, b_1\}$	$\{a_0, a_1, b_0, c_0, c_1\}$	Fig. 4.15(c)
Fig. 4.4(f)	$K_3$	$\{a_0, b_0, b_1\}$	$\{a_0, a_1, b_0, b_1\}$	Fig. 4.15(d)
Fig. 4.4(f)	$K_4$	$\{a_0, a_1, b_0, b_1\}$	$\{a_0, a_1, c_0, c_1\}$	Fig. 4.15(c)
Fig. 4.4(g)	$K_2$	$\{b_0, b_1\}$	$\{a, b_0, b_1, b_3\}$	Fig. 4.15(b)
Fig. 4.4(h)	$K_1$	$\{a_0\}$	$\{a_0, a_1, b_0, b_1, c_0\}$	Fig. 4.15(b)
Fig. 4.4(h)	$K_2$	$\{a_0, c_0\}$	$\{a_0, b_0, c_0, c_1\}$	Fig. 4.15(b)
Fig. 4.4(h)	$K_3$	$\{a_0, b_0, c_0\}$	$\{a_0, b_0, c_0, c_1\}$	Fig. 4.15(b)
Fig. 4.4(i)	$K_1$	$\{a_0\}$	$\{a_0, a_1, b_0, b_1, b_2\}$	Fig. 4.15(d)
Fig. 4.4(i)	$K_2$	$\{a_0, b_0\}$	$\{a_0, b_0, b_1, b_2\}$	Fig. 4.15(c)
Fig. 4.4(i)	$K_3$	$\{a_0, b_0, b_1\}$	$\{a_0, b_0, b_1, b_2\}$	Fig. 4.15(d)
Fig. 4.4(i)	$K_4$	$\{a_0, b_0, b_1, b_2\}$	$\{a_1, b_0, b_1, b_2\}$	Fig. 4.15(d)

**Table 4.4:** Relation between Fig. 4.4 and Fig. 4.15 in a loop-linked graph.

(2) The graph  $G$  does not contain any graph displayed in Figs. 4.12(a), 4.13(b), 4.13(c) or 4.15 as a vertex induced subgraph.

*Proof.* (1)  $\Rightarrow$  (2): This is guaranteed by the relation between the graph classes  $\mathcal{G}$  and  $\mathcal{S}$  in Table 4.5.

(2)  $\Rightarrow$  (1): Assume that the loop-linked connected graph  $G$  does not contain any graph displayed in Figs. 4.12(a), 4.13(b), 4.13(c) and 4.15 as a vertex induced subgraph. By the relation between graph classes  $\mathcal{G}$  and  $\mathcal{H}$  in the Table 4.5,  $G$  does not contain any graph in Figs. 4.12 and 4.13 as a vertex induced subgraph. By Lemma 4.35,  $G$  cannot have any graph in Fig. 4.4 as a vertex induced subgraph, either.  $\square$

**Lemma 4.37.** *Let  $G$  be a loop-linked reduced graph with. Then the following two statements are equivalent.*

(1) The graph  $G$  does not contain any graph displayed in Figs. 4.12(a), 4.13(b), 4.13(c) or 4.15 as a vertex induced subgraph.

$\mathcal{G}$	$\mathcal{H}$	$\mathcal{I}$
Fig. 4.15(a)	Figs. 4.12(b), 4.13(a)	Figs. 4.4(a), 4.12(b), 4.13(a)
Fig. 4.15(b)	Figs. 4.12(e), 4.13(d)	Figs. 4.4(b), 4.12(e), 4.13(d)
Fig. 4.15(c)	Figs. 4.12(c), 4.13(f)	Figs. 4.4(c), 4.4(d), 4.4(e), 4.12(c), 4.13(d), 4.13(f)
Fig. 4.15(d)	Fig. 4.13(g)	Figs. 4.4(i), 4.13(g)
Fig. 4.15(e)	Figs. 4.12(d), 4.12(f), 4.13(e)	Figs. 4.4(b), 4.12(d), 4.12(e), 4.12(f), 4.13(d), 4.13(e)

**Table 4.5:** Graph class  $\mathcal{H}$  is a subset of graph class  $\mathcal{G}$ . Every graph in graph class  $\mathcal{G}$  has a vertex induced subgraph in the graph class  $\mathcal{I}$ .

(2) *The graph  $G$  does not contain any graph in Fig. 4.2 as a vertex induced subgraph.*

*Proof. TODO: Detail and figure*

(1)  $\Rightarrow$  (2): Every graph demonstrated in Fig. 4.2 contains one of the graphs given in Figs. 4.12(a), 4.13(b), 4.13(c) and 4.15 as a vertex induced subgraph.

(2)  $\Rightarrow$  (1): Our task reduces to showing that a graph  $G$  will have one graph in Fig. 4.2 as a vertex induced subgraph on the condition that it is reduced and possesses a graph from Fig. 4.12(a) or 4.15(a) as a vertex induced subgraph.

We first consider the case that Fig. 4.12(a) is a vertex induced subgraph of  $G$ . Suppose that  $G[u, v]$  is isomorphic to Fig. 4.12(a). As  $G$  is reduced, there exists a vertex  $w$  such that  $N_G(w) \cap \{u, v\}$  is a singleton set. Therefore,  $G[u, v, w]$  belongs to the graph class Fig. 4.2(d) if  $w \in L_G$  and lies in Fig. 4.2(c) otherwise, as was to be shown.

Let the graphs described by Fig. 4.15(a) be  $\{H_0, H_1, \dots\}$  where  $H_i$  is the graph there with  $i + 3$  vertices (namely the dashed line stands for a path of length  $i$ ) for any nonnegative integer  $i$ . To finish the proof, we consider the remaining case that there exists a smallest nonnegative integer  $n$  such that we can find a vertex induced  $H_n$  in  $G$ .

Let  $V_{H_n} = \{u, v, x_0, \dots, x_n\}$ , where  $u, v$  are the unique pair of twin vertices in  $H_n$ ,  $x_0$  be the loop vertex and  $x_0, x_1, \dots, x_n$  be the path shown by the dashed line. Since  $u \approx v$  in  $G$ , there must be a vertex  $w$  of  $G$  fulfilling  $|N_G(w) \cap \{u, v\}| = 1$ . Without loss of generality, assume that  $uw \in E_G$  and  $vw \notin E_G$ . Let  $K = G[V_H \cup \{w\}]$ . The plan is to enumerate all the possible cases below, so that in each case we either identify a graph from Fig. 4.2 to be a vertex induced subgraph of  $G$ , or show that we can find a vertex induced  $H_i$  for some  $i < n$ , and thus get to a contradiction with the minimality of  $n$  (and so that case indeed should be impossible to happen).

Let  $S = \{i \mid wx_i \in E_K\}$ .

**Case 1:**  $w \in L_K$ .

**Case 1.1:**  $S = \emptyset$ .

The graph  $K$  belongs to the graph class displayed in Fig. 4.4(c).

**Case 1.2:**  $S = \{n\}$ .

The graph  $K - u$  belongs to the graph class displayed in Fig. 4.4(c).

**Case 1.3:**  $S \neq \emptyset, \{n\}$ .

The graph  $K[x_0, \dots, x_{\min S}, w, u]$  belongs to the graph class displayed in Fig. 4.4(c).

**Case 2:**  $w \notin L_K$ .

**Case 2.1:**  $S = \emptyset$ .

The graph  $K$  belongs to the graph class displayed in Fig. 4.4(a).

**Case 2.2:**  $S = \{n\}$ .

The graph  $K$  belongs to the graph class displayed in Fig. 4.4(b).

**Case 2.3:**  $\min S \leq n - 1, \min S + 1 \notin S$ .

The graph  $K[x_0, \dots, x_{k+1}, w]$  is isomorphic to  $H_k$  in Fig. 4.15(a), contradicting with our minimality assumption on  $n$ .

**Case 2.3:**  $\min S + 1 \in S$ .

**Case 2.3.1:**  $\min S = n - 1$ .

The graph  $K$  belongs to the graph class displayed in Fig. 4.4(e).

**Case 2.3.2:**  $\min S = n - 2$ .

If  $n \in S$ , then the graph  $K$  belongs to the graph class displayed in Fig. 4.4(g), otherwise the graph  $K - u$  belongs to the graph class displayed in Fig. 4.4(g).

**Case 2.3.3:**  $\min S \leq n - 3$ .

If there exists  $j \in \{\min S + 2, \dots, n - 1\} \cap S$ , then the graph  $K[x_0, \dots, x_{\min S}, w, u, x_j]$  is a copy of  $H_{\min S + 1}$ , violating our assumption on  $n$ .

**Case 2.3.1:**  $S = \{\min S, \min S + 1\}$

The graph  $K$  belongs to the graph class displayed in Fig. 4.4(g).

**Case 2.3.1:**  $S = \{\min S, \min S + 1, n\}$

The graph  $K - u$  belongs to the graph class displayed in Fig. 4.4(g). □

**Theorem 4.38.** *Let  $G$  be a loop-linked graph. The following statements are equivalent.*

- (1) The graph  $G$  is a line graph.
- (2) The graph  $G/\sim$  is a line graph.
- (3) The graph  $G/\sim$  is an ordinary line graph.
- (4) The graph  $G/\sim$  has an edge clique partition.
- (5) The graph  $G/\sim$  does not contain any graph displayed in Figs. 4.4, 4.12 or 4.13 as a vertex induced subgraph.
- (6) The graph  $G/\sim$  does not contain any graph in Fig. 4.2 as a vertex induced subgraph.
- (7) The graph  $G$  does not contain any graph in Fig. 4.2 as a vertex induced subgraph.
- (8) The graph  $G/\sim$  has a unique clique partition.
- (9) The graph  $G/\sim$  has a unique root graph.

*Proof.* (1)  $\Leftrightarrow$  (2): Lemma 4.7.

(2)  $\Leftrightarrow$  (3): Lemma 4.8.

(3)  $\Leftrightarrow$  (4): Theorem 4.27.

(4)  $\Leftrightarrow$  (5): Lemma 4.34.

(5)  $\Leftrightarrow$  (6): Lemmas 4.36 and 4.37.

(6)  $\Leftrightarrow$  (7): All graphs in Fig. 4.2 are reduced graphs.

(3)  $\Leftrightarrow$  (8): Theorem 4.32.

(8)  $\Leftrightarrow$  (9): Theorem 4.27. □

*Proof of Theorem 4.2.* Equivalence of (1) and (7) in Theorem 4.38. □

#### 4.12. Yet another operation $\boxminus$

**Lemma 4.39.** *If  $G$  is a line graph and  $v$  is a loop vertex of  $G$ , then  $G \boxminus v$  is a line graph.*

*Proof.* Suppose  $G$  is the line graph of a multigraph  $H$ . Let  $L = \partial_H^{-1}(\partial_H(v)) \setminus \{v\}$ . Take two disjoint sets  $L_0, L_1$  which are disjoint with  $V_H$  and we insist that there are bijections  $\gamma_i$  from  $L$  to  $L_i$  for  $i \in \mathbb{F}_2$ . Construct a multigraph  $H'$  with  $V_{H'} = V_H \cup \gamma_0(L) \cup \gamma_1(L)$ ,  $E_{H'} = E_H \setminus \{v\}$ ,  $\partial_{H'}(e) = \partial_H(e) \setminus \{v\}$  for  $e \in E_{H'} \setminus L$  and  $\partial_{H'}(l) = \{\gamma_0(l), \gamma_1(l)\}$  for  $l \in L$ . We have  $G \boxminus v = \mathfrak{L}(H')$ . □

**Remark 4.40.** (i) Unlike operators  $-$  and  $\hat{\cdot}$ , ordinary line graphs are not closed under the operator  $\boxminus$ .

(ii)  $\text{rank } \mathbb{A}_D = \text{rank } \mathbb{A}_{D \boxminus v} + 1$ .

(iii)  $L_D + L_{D \boxminus v} = N_D^-(v) \cap N_D^+(v)$ .

*Proof of Theorem 4.5.* Lemmas 4.8 and 4.39 state that, line graphs are closed under  $-$  and  $\boxminus$ . But, Figs. 4.3(a) and 4.3(b) are not line graphs.

For a connected loop-linked non-line graph, by Theorem 4.38, we can find a vertex induced subgraph  $H$  in Fig. 4.2. Every graph in Fig. 4.2 can be reduced to either Fig. 4.3(a) or Fig. 4.3(b) by a sequence of  $\boxminus$  through connected loop-linked graphs. □

## 5.

### 5.1. Critical subdigraphs

**Lemma 5.1.** *Let  $G$  be a graph and let  $S$  be a subset of  $V_G$ . If  $\text{rank } N_{G \ominus S} = \text{rank } N_G$ , then  $\text{rank } N_G = \text{rank } N_{G-S}$ .*

*Proof.* Since  $\text{rank } N_{G \ominus S} = \text{rank } N_G$ , we have

$$\text{Im } N_G = \{N_G(x) \mid x \subseteq V_G \setminus S\}. \quad (5.1)$$

It is easy to see that  $\text{Im } N_{G-S} = \{N_G(x) \setminus S \mid x \subseteq V_G \setminus S\}$ . To finish the proof, it suffices to show that the binary linear map  $f : \text{Im } N_G \rightarrow \text{Im } N_{G-S}$  given by  $f(x) = x \setminus S$  for every  $x \in \text{Im } N_G$  is a bijective map. The fact that  $f$  is onto is trivial and so we turn to the injectivity of  $f$ . Take  $x \in V_G \setminus S$  such that  $N_G(x) \setminus S = \emptyset$ . For every  $y \in 2^{V_G}$ , Eq. (5.1) allows us to pick  $z \subseteq V_G \setminus S$  such that  $N_G(y) = N_G(z)$ , and hence

$$\begin{aligned} & \langle y, N_G(x) \rangle \\ \equiv & \langle x, N_G(y) \rangle & (G \text{ is symmetric}) \\ \equiv & \langle x, N_G(z) \rangle & (N_G(y) = N_G(z)) \\ \equiv & \langle z, N_G(x) \rangle & (G \text{ is symmetric}) \\ \equiv & 0 & (N_G(x) \setminus S = \emptyset, z \subseteq V_G \setminus S) \end{aligned}$$

follows. This shows that  $N_G(x) = \emptyset$ , implying that  $f$  is a binary linear isomorphism between  $\text{Im } N_G$  and  $\text{Im } N_{G-S}$ , as desired. □

Given two graphs  $H$  and  $G$ , we will write  $H \propto G$  to mean that  $H$  is a vertex induced subgraph of  $G$ . Given two graph classes  $\mathcal{H}$  and  $\mathcal{G}$ , we similarly adopt the notation  $\mathcal{H} \propto \mathcal{G}$  for the fact that every graph  $G \in \mathcal{G}$  has a graph  $H \in \mathcal{H}$  as a vertex induced subgraph.

**Theorem 5.2.** *Let  $G$  be a connected graph and  $H$  be a connected nonsingular vertex induced subgraph of  $G$ . Then there exists a connected critical subgraph  $K$  satisfying  $H \propto K \propto G$ .*

*Proof.* Let  $H = G - R$  for some  $R \subseteq V_G$ . Since  $H$  is nonsingular, it holds  $\text{rank } N_{G-R} = \text{rank } N_{G \ominus R} = |V_G \setminus R|$ . Let  $\mathcal{C}$  be the set of elements  $T$  from  $2^R$  such that  $G - T$  is connected and  $\text{rank } N_{G \ominus T} = |V_G \setminus T|$ . Note that  $R \in \mathcal{C}$  and so we can find an inclusion minimal element  $S$  in  $\mathcal{C}$ . If  $\text{rank } N_G = \text{rank } N_{G \ominus S}$ , then we can apply Lemma 5.1 to yield  $\text{rank } N_{G-S} = \text{rank } N_G = \text{rank } N_{G \ominus S} = |V_G \setminus S|$  and hence we can just choose  $K$  to be  $G - S$ .

It suffices to show  $\text{rank } N_G = \text{rank } N_{G \ominus S}$ . To obtain a contradiction, suppose  $\text{rank } N_G > \text{rank } N_{G \ominus S}$  and so we can find  $v \in S$  such that  $N_G(v) \notin \text{Im } N_{G \ominus S}$ . Find a shortest path from  $V_G \setminus S$  to  $v$ , say  $v_1 \cdots v_k$  where  $v_1 \notin S$  and  $v_k = v$ . Note that  $v \in S$  and so  $k \geq 2$  holds. It is easy to check that  $S \setminus \{v_2\} \in \mathcal{C}$ , violating the minimality of  $S$ .  $\square$

**Remark 5.3.** *This proof works for general fields.*

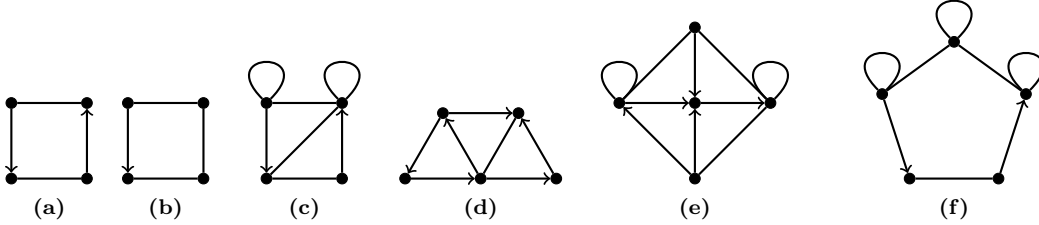
*Proof of Theorem 3.4.* Suppose that a graph  $G$  lies in a class  $\mathcal{C}$ .

- (i) Let  $H$  be a connected subgraph of  $G$  with two vertices.
- (ii) Take a loop vertex  $v$  of  $G$  and let  $H = G[v]$ .
- (iii) Let  $H$  be a subgraph of  $G$  which is isomorphic to one in Fig. 4.1, as guaranteed by Theorem 3.3.
- (iv) Let  $H$  be a subgraph of  $G$  which is isomorphic to one in Fig. 4.2  $G$ , as guaranteed by Theorem 3.3.

The results follow from Theorem 5.2.  $\square$

**Remark 5.4.** *Critical subgraphs of a graph  $G$  are same with critical subgraphs of  $G / \sim$ .*

**Example 5.5.** *Strongly connected digraphs in Fig. 5.1 do not have strongly connected critical subdigraphs. subdigraph or subgraph? do we need to define subgraph?*



**Figure 5.1:** Some strongly connected digraphs which do not have strongly connected critical subdigraphs.

**Problem 5.6.** *Why do most strongly connected digraphs have strongly connected critical subdigraphs? Is there any good characterization for strongly connected digraphs which have strongly connected critical subdigraphs?*

**Lemma 5.7.** *For a digraph  $D$  and a critical subdigraph  $E$  of  $D$ , let  $T = V(D) \setminus V(E)$ . The phase space  $\mathcal{PS}(D \ominus T)$  is disjoint union of  $\mathcal{PS}(D \ominus T)[x + \mathfrak{C}(D \ominus T)]$  where  $x \in \mathbb{F}_2^{V(D)|T}$ . In particular,  $\mathbb{F}_2^{V(D)}$  is the disjoint union of  $x + \mathfrak{C}(D)$ , where  $x \in \mathbb{F}_2^{V(D)|T}$ .*

*Proof.* Easy.  $\square$

**Remark 5.8.** *This lemma allows us to extend the results on binary code to the results on shifted binary code.*

## 5.2. Multi-mixed-graphs

**All the notations in this section is compatible with the notation in the Introduction.**

Given a sign  $\circ \in \{+, -\}$ , the  $\circ$ -coboundary map of a mixed graph  $M$ , denoted by  $d_M^\circ$ , is a map from  $V(M)$  to  $F(M)$ , which maps  $v$  to  $\{f \in F(M) \mid v \in \partial_M^\circ(f)\}$ .

The non-loop-edge-degree matrix of a mixed graph  $M$ , denoted by  $D(M)$ , is a diagonal matrix in  $\mathbb{F}_2^{V(M) \times V(M)}$ , whose  $v, v$ -entry is  $|(d_M^-(v) \cap d_M^+(v)) \setminus L(M)|$ .

The Laplacian matrix of a mixed graph  $M$ , denoted by  $\Delta(M)$ , is an element of  $\mathbb{F}_2^{V(M) \times V(M)}$ , whose  $v, w$ -entry is  $|d_M^-(v) \cap d_M^+(w)|$ . **Our definition is different from that used in [? ].**

The adjacency matrix of a mixed graph  $M$ , denoted by  $\mathbb{A}(M)$ , is  $D(M) - \Delta(M)$ .

Let  $\text{Ker } \mathbb{B}^\circ(M)^\top$  stand for the subspace  $\{x \in \mathbb{F}_2^{V(M)} \mid \mathbb{B}^\circ(M)^\top x = \mathbf{0}\}$ .

**Lemma 5.9.** *Let  $M$  be a strongly connected mixed graph. For every sign  $\circ \in \{+, -\}$ , it holds*

$$\text{Ker } d_M^\circ = \begin{cases} \{\emptyset\}, & \text{if } A_M \neq \emptyset, \\ \{\emptyset, V_M\}, & \text{if } A_M = \emptyset. \end{cases}$$

*In particular, if  $A_M \neq \emptyset$ , then  $\text{rank } d_M^\circ = |V_M|$  and  $d_M^\circ$  is an injective map.*

*Proof.* For any  $x \in \text{Ker } d_M^\circ$  and any  $e \in E_M$ , it follows from Eq. (2.1) that

$$0 = \langle \emptyset, e \rangle = \langle d_M^\circ(x), e \rangle = \langle x, \partial_M^\circ(e) \rangle = \sum_{v \in \partial_M^\circ(e)} x^*(v). \quad (5.2)$$

This implies that  $x^*$  takes constant value on each component of  $M'$  where  $M'$  is the multigraph with  $V_{M'} = V_M$ ,  $E_{M'} = E_M$  and  $d_{M'}^\circ = d_M^\circ|_{E_M}$  for  $\circ \in \{+, -\}$ . If  $A_M = \emptyset$ , we have  $M = M'$  and so  $\text{Ker } d_M^\circ = \{\emptyset, V_M\}$  follows. Now consider the case that  $A_M \neq \emptyset$ . As  $M$  is strongly connected and  $A_M \neq \emptyset$ , each component  $C$  of  $M'$  contains a vertex  $v$  such that  $\{v\} = \partial_M^\circ(e)$  for some  $e \in A_M$ . In view of Eq. (5.2), for each  $x \in \text{Ker } d_M^\circ$  we have  $x^*(v) = 0$ . Combining with the fact that  $x^*$  takes constant value on each component of  $M'$ , this gives  $\text{Ker } d_M^\circ = \{\emptyset\}$ .  $\square$

**Remark 5.10.** *Let  $\mathbb{F}$  be a field and let  $M$  be a strongly connected  $M$ -graph. For any sign  $\circ \in \{+, -\}$ , generalizing the case of  $\mathbb{F} = \mathbb{F}_2$ , we may view the  $(0, 1)$  matrix  $\mathbb{B}^\circ(M)$  as a matrix over  $\mathbb{F}$ . The proof of Lemma 5.9 follows the proof of [?, Theorem 8.2.1]. Let us mention the following general observations: If the characteristic of  $\mathbb{F}$  is 2, then over  $\mathbb{F}$  we have*

$$\text{Ker } \mathbb{B}^\circ(M)^\top = \begin{cases} \{\mathbf{0}\}, & A_M \neq \emptyset, \\ \{t \mathbf{1} \mid t \in \mathbb{F}\}, & A_M = \emptyset. \end{cases}$$

*If the characteristic of  $\mathbb{F}$  is not 2, then over  $\mathbb{F}$  we have*

$$\text{Ker } \mathbb{B}^\circ(M)^\top = \begin{cases} \{\mathbf{0}\}, & \text{if } A_M \neq \emptyset \text{ or } M[E_M] \text{ contains an odd cycle,} \\ \{t(\chi_A - \chi_B) \mid t \in \mathbb{F}\}, & \text{if } H \text{ is a bipartite multigraph with partite sets } A \text{ and } B. \end{cases}$$

*Note that  $\text{rank } \mathbb{B}^\circ(M)$  over a field of characteristic 2 is always no bigger than  $\text{rank } \mathbb{B}^\circ(M)$  over a field of characteristic not equal to 2.*

### 5.3. Line digraphs of mixed multigraphs

The next lemma puts [?, Fact 22] in a bit more general context. However, it is basically already known by Doob [? ? ].

**Lemma 5.11.** *Let  $M$  be a strongly connected mixed graph. Then we have*

$$\text{rank } N_{\mathcal{L}(M)} = \begin{cases} |V_M|, & \text{if } A_M \neq \emptyset, \\ |V_M| - 1, & \text{if } A_M = \emptyset \text{ and } |V_M| \text{ is odd,} \\ |V_M| - 2, & \text{if } A_M = \emptyset \text{ and } |V_M| \text{ is even.} \end{cases}$$

*Proof.* When  $A_M \neq \emptyset$ , Lemma 5.9 tells us that  $\text{rank } d_M^- = \text{rank } \partial_M^+ = |V_M|$  and then it follows from Sylvester's rank inequality and Eq. (2.2) that

$$\text{rank } N_{\mathcal{L}(M)} = \text{rank}(d_M^- \partial_M^+) \geq \text{rank } d_M^- + \text{rank } \partial_M^+ - |V_M| = |V_M|,$$

and

$$\text{rank } N_{\mathcal{L}(M)} = \text{rank}(d_M^- \partial_M^+) \leq \min\{\text{rank } d_M^-, \text{rank } \partial_M^+\} = |V_M|.$$

From now on, we assume  $A_M = \emptyset$ , namely  $M$  is indeed a multigraph without loops.

Suppose that  $M$  has a cycle, say  $e_1, \dots, e_m$ , which must have length  $m \geq 2$ . Then  $\sum_{i=1}^m N_{\mathcal{L}(M)}(e_i) = \emptyset$  and hence  $\text{rank } N_{\mathcal{L}(M)} = \text{rank } N_{\mathcal{L}(M) \ominus e_1}$ . By Lemma 5.1, we obtain  $\text{rank } N_{\mathcal{L}(M)} = \text{rank } N_{\mathcal{L}(M) - e_1}$ .

The above discussion tells us that we can remove zero or more edges from  $M$  to obtain a spanning tree  $T$  of  $M$  which satisfies

$$\text{rank } N_{\mathcal{L}(M)} = \text{rank } N_{\mathcal{L}(T)}.$$

In view of this, we may assume that  $M$  is a tree.

For any two vertices  $u$  and  $v$  in  $M$ , letting  $P$  be the set of edges on the unique/shortest path of  $M$  connecting  $u$  and  $v$ , we can find that  $\sum_{e \in P} \partial_M(e) = u + v$ . Since  $M$  is loopless, it thus follows that, for each  $S \subseteq V_M$ ,  $S$  lies in  $\text{Im } \partial_M$  if and only if  $|S|$  is even. To proceed, we also note that, as a consequence of Lemma 5.9, we have

$$\text{Ker } d_M = \{\emptyset, V_M\} \quad (5.3)$$

and

$$\text{rank } \partial_M = \text{rank } d_M = |V_M| - 1. \quad (5.4)$$

**Case 1:**  $|V_M|$  is odd and so  $V_M \notin \text{Im } \partial_M$ .

$$\begin{aligned}
\text{rank } N_{\mathfrak{L}(M)} &= \dim \text{Im } N_{\mathfrak{L}(M)} \\
&= \dim \text{Im } d_M^- \partial_M^+ \quad (\text{Eq. (2.2)}) \\
&= \dim \text{Im } \partial_M^+ \quad (V_M \notin \text{Im } \partial_M \text{ and Eq. (5.3)}) \\
&= \text{rank } \partial_M^+ \\
&= |V_M| - 1. \quad (\text{Eq. (5.4)})
\end{aligned}$$

**Case 2:**  $|V_M|$  is even and so  $V_M \in \text{Im } \partial_M$ .

$$\begin{aligned}
\text{rank } N_{\mathfrak{L}(M)} &= \dim \text{Im } N_{\mathfrak{L}(M)} \\
&= \dim \text{Im } d_M^- \partial_M^+ \quad (\text{Eq. (2.2)}) \\
&= \dim \text{Im } \partial_M^+ - 1 \quad (V_M \in \text{Im } \partial_M \text{ and Eq. (5.3)}) \\
&= \text{rank } \partial_M^+ - 1 \\
&= |V_M| - 2. \quad (\text{Eq. (5.4)})
\end{aligned}$$

□

#### 5.4. Critical subgraphs of line graphs

*Proof of Theorem 3.5.* A corollary of Lemma 5.11.

□

**Corollary 5.12.** *Transversal of all connected critical subgraphs of a line graph. TODO*

#### 5.5. Generalization of matrix-tree theorem

In this section, all the calculation is done over  $\mathbb{Z}$ .

First, we need to define the incidence matrix over  $\mathbb{Z}$ .

$$\Delta(M) = D(M) - A(M)$$

An *orientation*  $O$  of a mixed graph  $M$  is a multidigraph, which is defined on the same vertex set, equipped with a mapping  $o: F(M) \rightarrow F(O)$  that satisfies for every  $f \in F(M)$ ,  $\partial_O^\circ(o(f)) \subseteq \partial_M^\circ(f)$  for  $\text{sign } \circ \in \{+, -\}$  and  $\bigcup_{\circ \in \{+, -\}} \partial_O^\circ(o(f)) = \bigcup_{\circ \in \{+, -\}} \partial_M^\circ(f)$ .

Let  $M$  be a mixed graph and  $O$  be an orientation of  $M$ .

$$\mathbb{B}_O^\circ(M) = \text{sgn}(\circ)(2\mathbb{B}^\circ(O) - \mathbb{B}^\circ(M))$$

$$\mathbb{B}_O^\circ(M)(v, f) = \begin{cases} \text{sgn}(\circ), & v \in \partial_O^\circ(f), \\ -\text{sgn}(\circ), & v \in \partial_M^\circ(f), v \notin \partial_O^\circ(f), \end{cases}$$

$$\Delta(M) = \mathbb{B}_O^+(M)^\top \mathbb{B}_O^-(M)$$

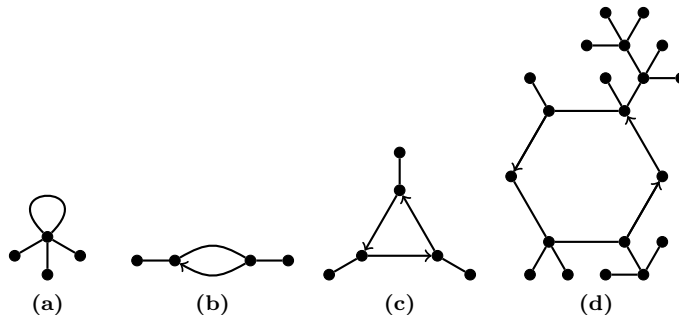
A *functional* digraph is a digraph where out-degree of every vertex is 1.

An *injective* digraph is a digraph where in-degree of every vertex is 1.

A *delicious* mixed graph is a mixed graph that has a unique functional orientation, and a unique injective orientation.

**what is a cycle?** The sign of a delicious mixed graph  $M$ , denoted by  $\text{sgn}(M)$ , is

$$\prod_{C \text{ is a cycle of } M} (-1)^{|V(C)|-1}.$$



**Figure 5.2:** TODO



**Theorem 5.13.** For a mixed graph  $M$ ,

$$\det \Delta(M) = (-1)^{|V(M)|} \sum_{\text{Spanning delicious sub-M-graph } D \text{ of } M} \text{sgn}(D)$$

where the determinant is calculated over  $\mathbb{Z}$ .

*Proof.* Using Eq. (2.5) and the Binet-Cauchy formula,

$$\begin{aligned} & \det \Delta(M) \\ &= \det \mathbb{B}_O^-(M) \mathbb{B}_O^+(M)^\top \\ &= \sum_{S \text{ is spanning subgraph of } M} \mathbb{B}_O^+(M[S])^\top \mathbb{B}_O^-(M[S]) \\ &= (-1)^{|V(M)|} \sum_{\text{Spanning delicious sub-M-graph } D \text{ of } M} \text{sgn}(D). \end{aligned}$$

□

**Remark 5.14.** We can obtain all principle minors matrix-tree theorem [?] by adding loops.

**Theorem 5.15.** Generalize to vector bundle.

## 6. Euler characteristics and gem index

### 6.1. Euler characteristics

For a graph  $G$ , the *Euler characteristic* of  $G$  over  $\mathbb{F}_2$ , denoted by  $\chi(G)$ , is  $|V_G| - |E_G|$ . The *Euler form* of a graph  $G$  over  $\mathbb{F}_2$ , denoted by  $\chi_G$ , is a binary quadratic form on  $2^{V_G}$  given by  $\chi_G(x) = \chi(G[x])$ . **(TODO: define induced graph)**

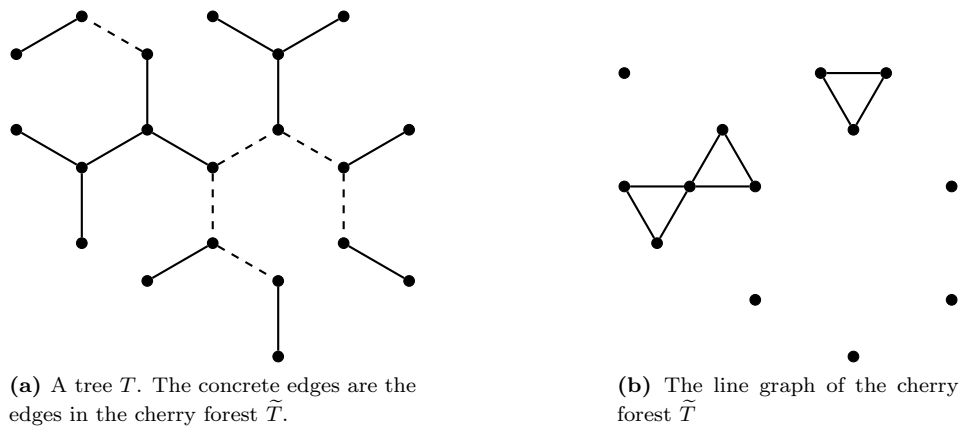
**Remark 6.1.** (1) For each  $v \in V_G$ ,  $\chi_G(v) \equiv 0$  if  $v$  is a loop vertex, and  $\chi_G(v) \equiv 1$  if  $v$  is not a loop vertex. **(TODO: define loop vertex)**

(2) For every  $x, y \subseteq V_G$ , it holds  $\chi_G(x) + \chi_G(y) + \chi_G(x + y) \equiv \mathbb{A}(x, y) + |x \cap y \cap L_G|$ . In particular, if  $G$  is loopless,  $\chi_G(x) + \chi_G(y) + \chi_G(x + y) \equiv \mathbb{A}(x, y)$ .

(3) For a forest  $F$ ,  $\chi(F)$  is the number of components of  $F$ .

### 6.2. Cherry forest

Let  $T$  be a tree with an even number of vertices. The *cherry forest* of  $T$ , denoted by  $\tilde{T}$ , is the spanning forest obtained from  $T$  by removing those edges whose deletion cause an even component **(TODO: a component of even number of vertices)**. Fig. 6.1 gives an example.



**Figure 6.1:** TODO

**Theorem 6.2.** For every tree  $T$  with an even number of vertices, it holds  $\chi(\mathfrak{L}(\tilde{T})) \equiv |V_T|/2$ .

*Proof.* **(TODO: induction. Cherry.  $P_2$ . A figure)**

□

### 6.3. Marble graphs

A loopless graph  $G$  is a *marble* graph if there exists quadratic form  $q_G$  on  $\mathfrak{C}_G$  such that  $q_G \circ N_G = \chi_G$ . We give an equivalent definition.

**Lemma 6.3.** *For a loopless graph  $G$ , it is a marble graph if and only if  $\text{Ker } N_G \subseteq \text{Ker } \chi_G$ .*

*Proof.* The necessity is trivial. We only need to consider the sufficiency.

For every  $x, y \subseteq 2^{V_G}$ ,

$$q_G(N_G(x)) + q_G(N_G(y)) = \chi_G(x) + \chi_G(y) = \chi_G(x + y) + \mathbb{A}_G(x, y).$$

For every  $x, y \subseteq 2^{V_G}$  such that  $x + y \in \text{Ker } N_G \subseteq \text{Ker } \chi_G$ ,

$$q_G(N_G(x)) + q_G(N_G(y)) = \chi_G(x + y) + \mathbb{A}_G(x, y) = 0.$$

Therefore,  $q_G$  is a well-defined quadratic form. □

Here is a quick corollary.

**Corollary 6.4.** *All loopless nonsingular graphs are marble. In particular, critical subgraphs of loopless graphs are marble.*

**Problem 6.5.** *Is there any graph property that tells whether a graph is marble?*

We give an answer for line graphs in §6.6. The answer is unknown in general.

### 6.4. Ruby index

In this section, assume that  $G$  is a loopless graph,  $H$  is a critical subgraph of  $G$ , and set  $\overline{V_H} = V_G \setminus V_H$ .

The *ruby index*  $\theta_{G,H}$  is a linear map from  $2^{V_G}$  to  $\mathbb{F}_2$  defined by setting

$$\theta_{G,H}(N_G(x) + y) \equiv \chi_G(x)$$

for every  $x \subseteq V_H$  and  $y \subseteq \overline{V_H}$ .

Map  $\varphi_{G,H} : V_G \rightarrow 2^{V_G}$  is the unique map such that  $N_G \circ \varphi_{G,H} = 0$  and  $\text{Im}(\varphi_{G,H} + \text{Id}) \subseteq 2^{V_H}$ . Let  $\psi_{G,H} \equiv \chi_G \circ \varphi_{G,H}$ . Clearly, for  $v \in V_H$ ,  $\psi_{G,H}(v) = \emptyset$  and hence  $\psi_{G,H}(v) \equiv 0$ .

Let

$$\mathfrak{B}_{G,H} = \sum_{v \in V_G} \psi_{G,H}(v)v + \mathfrak{C}_G.$$

**(TODO: Later, we will show that  $\mathfrak{B}_{G,H}$  does not depend on the choice of the critical subgraph  $H$ .)**

We omit the subscripts when they are clear from the content.

**Lemma 6.6.** *Let  $z_0, z_1 \in y + \mathfrak{C}_G$  where  $y \in \overline{V_H}$  and  $v \in V_G$ . If  $z_0 + z_1 = N_G(v)$ , then it holds*

$$\theta(z_0) + \theta(z_1) \equiv \psi(v) + v^*(y + z_0) + 1.$$

Furthermore, if  $z_0 \xrightarrow{w}_G z_1$  for some word  $w = w_0 \dots w_{k-1} \in V_G^*$ , then

$$\theta(z_0) + \theta(z_1) \equiv \sum_{i=0}^{k-1} (\psi(w_i) + w_i^*(x)).$$

*Proof.* Let  $x \in 2^{V_H}$  be the unique element that satisfies  $N_G(x) + y = z_0$ , then

$$z_1 = z_0 + N_G(v) = N_G(x + v) + y = N_G(x + v + \varphi(v)) + y.$$

We can proceed with the following calculation.

$$\begin{aligned} & \theta(z_0) + \theta(z_1) \\ \equiv & \chi_G(x) + \chi_G(x + v + \varphi(v)) \\ \equiv & \chi_G(v + \varphi(v) + \mathbb{A}(x, v) + \mathbb{A}(x, \varphi(v))) \\ \equiv & \chi_G(v) + \chi_G(\varphi(v)) + \mathbb{A}(x + \varphi(v), \varphi(v)) + \mathbb{A}(x, v) \\ \equiv & \psi(v) + \chi_G(v) + \mathbb{A}(x, v) \\ \equiv & \psi(v) + 1 + v^*(z_0 + y). \end{aligned}$$

□

### 6.5. Sapphire index

In this section, assume that  $G$  is the line graph of a connected multigraph  $K$ , and  $H$  is a critical connected line subgraph of  $G$ , and set  $\overline{V_H} = V_G \setminus V_H$ . The set  $\overline{V_H}$  induces a subgraph in  $K$  (**TODO: by theorem ??**), denoted by  $J$ . It is clear that,  $\mathfrak{L}(J) = H$ . (**TODO: a figure.**)

The *sapphire index*  $\tilde{\theta}_{G,H}$  is a linear map from  $2^{V_G}$  to  $\mathbb{Z}$  defined by setting

$$\tilde{\theta}_{G,H}(N_G(x) + y) = |\partial_J(x)|,$$

for every  $x \subseteq V_H$  and  $y \subseteq \overline{V_H}$ .

The image of  $\tilde{\theta}_{G,H}$  can be easily determined, and we have

$$\text{Im } \tilde{\theta}_{G,H} = \begin{cases} \{0, 1, \dots, |V_J|\}, & L_G \neq \emptyset, \\ \{0, 2, \dots, |V_J| - 1\}, & L_G = \emptyset. \end{cases}$$

Map  $\tilde{\psi}_{G,H} : V_G \rightarrow \mathbb{F}_2$  is defined by

$$\tilde{\psi}_{G,H}(v) \equiv \begin{cases} 0, & \partial_K(v) \subseteq V_J, \\ 1, & \partial_K(v) \not\subseteq V_J, \end{cases}$$

for  $v \in V_G$ . It is clear that, for  $v \in V_H$ ,  $\tilde{\psi}_{G,H}(v) \equiv 0$ .

Let

$$\tilde{\mathfrak{B}}_{G,H} = \sum_{v \in V_G} \tilde{\psi}_{G,H}(v)v + \mathfrak{C}_G.$$

We will show  $\tilde{\mathfrak{B}}_{G,H}$  does not depend on the choice of the critical subgraph  $H$  later.

We omit the subscripts when they are clear from the content.

**Lemma 6.7.** *Let  $z_0, z_1 \in y + \mathfrak{C}_G$  where  $y \in \overline{V_H}$ , and  $v \in V_G$ . If  $z_0 + z_1 = N_G(v)$ , then it holds:*

- (i)  $\tilde{\theta}(z_1) - \tilde{\theta}(z_0) = -1$  if  $v \in L_G$ ,  $v^*(y + z_0) = 1$ ;
- (ii)  $\tilde{\theta}(z_1) - \tilde{\theta}(z_0) = 1$  if  $v \in L_G$ ,  $v^*(y + z_0) = 0$ ;
- (iii)  $\tilde{\theta}(z_1) - \tilde{\theta}(z_0) = 0$  if  $v \notin L_G$ ,  $v^*(y + z_0) = 1$  and  $\tilde{\psi}(v) = 0$ ;
- (iv)  $\tilde{\theta}(z_1) - \tilde{\theta}(z_0) \in \{-2, 2\}$  if  $v \notin L_G$ ,  $v^*(y + z_0) = 0$  and  $\tilde{\psi}(v) = 0$ ;
- (v)  $\tilde{\theta}(z_1) + \tilde{\theta}(z_0) = |V_K|$  if  $v \notin L_G$ ,  $v^*(y + z_0) = 1$  and  $\tilde{\psi}(v) = 1$ ;
- (vi)  $\tilde{\theta}(z_1) + \tilde{\theta}(z_0) = |V_K| - 2$  if  $v \notin L_G$ ,  $v^*(y + z_0) = 0$  and  $\tilde{\psi}(v) = 1$ .

*Proof.* **TODO: proof** □

### 6.6. Loopless line graphs and marble graphs

**Lemma 6.8.** *Let  $G$  be the line graph of a connected loopless multigraph  $K$ , and  $H$  be a critical connected subgraph of  $G$ . We have*

$$\psi \equiv \frac{|V_K|}{2} \tilde{\psi}.$$

*Proof.* The set  $V_H$  induces a subgraph in  $K$ , denoted by  $J$ . Let  $v$  be a arbitrary vertex of  $G$ . Since the multigraph  $K$  is loopless,  $|\partial_K(v)| = 2$ . Assume that  $\partial_K(v) = \{s, t\}$ .

**Case 1:**  $\tilde{\psi}(v) \equiv 0$ .

The vertices  $s$  and  $t$  are both in  $J$ . Let  $P$  be the unique path in  $J$  connecting  $s$  and  $t$ . Then,  $\phi(v) = v + E_P$  (**Why?**), hence  $\psi(v) = 0$ .

**Case 2:**  $\tilde{\psi}(v) \equiv 1$ .

Recall the classification of  $J$ , exactly one of  $s$  and  $t$  is in  $J$ . Without loss of generality, assume that  $V_K = s + V_J$ . Adding the edge  $st$  to  $J$ , we obtain a spanning tree of  $K$ , denoted by  $T$ . Then,  $\phi(v) = E_{\tilde{T}}$  (**Why?**), where  $\tilde{T}$  is the cherry forest of  $T$ . Therefore, using Theorem 6.2, we have  $\psi(v) \equiv \chi_G(\phi(v)) \equiv \chi_G(E_{\tilde{T}}) \equiv \chi(G[E_{\tilde{T}}]) \equiv \chi_G(\mathfrak{L}(\tilde{T})) \equiv |V_T|/2 \equiv |V_K|/2$ . □

**Lemma 6.9.** *Let  $G$  be a loopless graph equipped with a critical subgraph. The following statements are equivalent.*

- (i)  $G$  is marble.
- (ii)  $\psi_G \equiv 0$ .
- (iii)  $\mathfrak{B}_G = \mathfrak{C}_G$ .

*Proof.* (i)  $\Leftrightarrow$  (ii): The set  $\psi_G^{-1}(1)$  is a basis of  $\text{Ker } N_G$ . And the result follows from Lemma 6.3.

(ii)  $\Leftrightarrow$  (iii): The result follows directly from definition. □

**Theorem 6.10.** *Let  $K$  be a connected loopless multigraph. Then,  $\mathfrak{L}(K)$  is marble if and only if  $|V_K| \not\equiv 2 \pmod{4}$ . In particular, if the line graph of a connected loopless multigraph is emerald, then it is marble.*

*Proof.* Combining Lemmas 6.8 and 6.9,  $\mathfrak{L}(K)$  is marble, if and only if, either  $\tilde{\psi}_{\mathfrak{L}(K)} \equiv 0$ , which is equivalent to  $|V_K| \equiv 1 \pmod{2}$ , or  $|V_K| \equiv 0 \pmod{4}$ . □

## 7. Lit-only core group

## 8. Gem index and structure of phase spaces

## 9. Lit-only group

## 10. Combinatorial approach to structure of phase spaces

**ATTENTION:** The zero  $\emptyset$  of vector space  $2^V$  is always denoted by  $\mathbf{0}$ , and  $\emptyset$  is only considered as empty set. Throughout this section, we always assume that  $D$  is a digraph and  $v$  is a vertex of  $D$ .

### 10.1. In $D \ominus v$

**Lemma 10.1.** *The following two statements hold:*

- (1) *If all elements of  $V_D$  can reach  $v$  in  $D$ , then for every  $\alpha \in 2^{V_D} \setminus \{\mathbf{0}\}$ , there exists  $\vec{\alpha} \in 2^{V_D}$  such that  $\alpha \rightarrow_{D \ominus v} \vec{\alpha}$  and  $v^*(\vec{\alpha}) = 1$ .*
- (2) *If  $v$  can reach every element of  $V_D$  in  $D$ , then for every  $\beta \in 2^{V_D} \setminus \{L_D\}$ , there exists  $\overleftarrow{\beta} \in 2^{V_D}$  such that  $\beta \leftarrow_{D \ominus v} \overleftarrow{\beta}$  and  $v^*(\overleftarrow{\beta} + L_D) = 1$ .*

*Proof.* (1): Since  $\alpha \neq \mathbf{0}$  and each  $w \in V_D$  can reach  $v$  in the digraph  $D$ , there is a shortest path  $w_0, w_1, \dots, w_k = v$  in  $D$  such that  $w_0$  is the only vertex  $x$  among  $w_0, \dots, w_k$  satisfying  $x^*(\alpha) = 1$ . It is easy to see that, for  $\vec{\alpha} = \alpha + \sum_{i=0}^{k-1} N_{D \ominus v}(w_i)$  we have  $\alpha \xrightarrow{w_0 \dots w_{k-1}}_{D \ominus v} \vec{\alpha}$  and  $v^*(\vec{\alpha}) = 1$ . Note that when  $k = 0$  we view  $w_0 \dots w_{k-1}$  as the empty word and thus  $\vec{\alpha} = \alpha$ .  
(2): Since  $\beta \neq L_D$ , the set  $S = \{x \mid x^*(L_D + \beta) = 1\}$  is nonempty. Therefore, as we assume that  $v$  can reach every  $w \in V_D$ , there is a shortest path  $v = w_0, \dots, w_k$  in  $D$  such that  $\{w_0, \dots, w_k\} \cap S = \{w_k\}$ . It is easy to see that for  $\overleftarrow{\beta} = \beta + \sum_{i=1}^k N_{D \ominus v}(w_i)$  we have  $\overleftarrow{\beta} \xrightarrow{w_1 \dots w_k}_{D \ominus v} \beta$  and  $v^*(\overleftarrow{\beta} + L_D) = 1$ . Note that when  $k = 0$  we view  $w_1 \dots w_k$  as the empty word and thus  $\overleftarrow{\beta} = \beta$ .  $\square$

### 10.2. Recurrent, normal

A coset  $\mathfrak{B}$  of  $\mathfrak{C}_D$  is *v-recurrent* if

$$\mathfrak{B} \setminus \{\mathbf{0}, v\} \rightarrow_D \mathfrak{B} \setminus \{L_D, L_D + v\}.$$

The digraph  $D$  is *v-recurrent*, if every coset of  $\mathfrak{C}_D$  is *v-recurrent*. Recall that, a coset  $\mathfrak{B}$  of  $\mathfrak{C}_D$  is *recurrent* if

$$\mathfrak{B} \setminus \{\mathbf{0}\} \rightarrow_D \mathfrak{B} \setminus \{L_D\}$$

The digraph  $D$  is *recurrent*, if every coset of  $\mathfrak{C}_D$  is recurrent.

The cosets  $\mathfrak{C}_D, \mathfrak{C}_D + L_D, \mathfrak{C}_D + v, \mathfrak{C}_D + v + L_D$  are *v-normal* if they satisfies

$$\begin{aligned} \mathfrak{C}_D \setminus \{v\} &\rightarrow_D \mathbf{0}, \\ L_D &\rightarrow_D (\mathfrak{C}_D + L_D) \setminus \{L_D + v\}, \\ \mathfrak{C}_D + v \setminus \{\mathbf{0}\} &\rightarrow_D v, \\ L_D + v &\rightarrow_D (\mathfrak{C}_D + L_D + v) \setminus \{L_D\}, \end{aligned}$$

respectively. The digraph  $D$  is *v-normal*, if the cosets  $\mathfrak{C}_D, \mathfrak{C}_D + L_D, \mathfrak{C}_D + v, \mathfrak{C}_D + L_D + v$  are *v-normal* and other cosets are recurrent. The digraph  $D$  is *normal*, if

$$\begin{aligned} \mathfrak{C}_D &\rightarrow \mathbf{0}, \\ L_D &\rightarrow \mathfrak{C}_D + L_D, \end{aligned}$$

and cosets other than  $\mathfrak{C}_D$  and  $\mathfrak{C}_D + L_D$  are recurrent.

### 10.3. Additional notations

**(TODO: distinguish  $*$  and  $\star$ )**

For  $a \in V_D$ ,  $x, y \in 2^{V_D}$ ,  $\mathfrak{B} \subseteq 2^{V_D}$  and  $W = w_1 w_2 \dots w_k \in V_D^*$ , we adopt the following notation:

- $x \xrightarrow{a}_{D \parallel \mathfrak{B}} y$ :  $x, y \in \mathfrak{B}$  and  $x \xrightarrow{a}_D y$ ;
- $x \xleftrightarrow{a}_D y$ : at least one of  $x \xrightarrow{a}_D y$  and  $y \xrightarrow{a}_D x$  holds;
- $x \xleftrightarrow{a}_{D \parallel \mathfrak{B}} y$ : at least one of  $x \xrightarrow{a}_{D \parallel \mathfrak{B}} y$  and  $y \xrightarrow{a}_{D \parallel \mathfrak{B}} x$  holds;
- $x \xrightarrow{W}_D y$ : there exist  $x_0, \dots, x_k \in 2^{V_D}$  such that  $x_{i-1} \xleftrightarrow{w_i}_D x_i$  for each  $i \in \{1, \dots, k\}$ ,  $x_0 = x$  and  $x_k = y$ ;
- $x \xleftrightarrow{W}_{D \parallel \mathfrak{B}} y$ : there exist  $x_0, \dots, x_k \in \mathfrak{B}$  such that  $x_{i-1} \xleftrightarrow{w_i}_{D \parallel \mathfrak{B}} x_i$  for each  $i \in \{1, \dots, k\}$ ,  $x_0 = x$  and  $x_k = y$ .

When writing  $x \xleftrightarrow{W}_D y$  or  $x \xleftrightarrow{W}_{D \parallel \mathfrak{B}} y$ , we mean  $x \xleftrightarrow{W}_D y$  or  $x \xleftrightarrow{W}_{D \parallel \mathfrak{B}} y$ , respectively, for some  $W \in V(D)^*$ .

#### 10.4. From $D - v$ to $D \ominus v$

We always write  $\mathbb{p}$  for the natural projection from  $2^{V_D}$  to  $2^{V_{D-v}}$  that sends  $\alpha$  to  $\alpha \setminus \{v\}$ .

A subset  $\mathfrak{B}$  of  $2^{V_D}$  is  $v$ -full, if  $\mathfrak{B} = \mathfrak{B} + v$ .

**Lemma 10.2.** *Let  $\mathfrak{B} \subseteq 2^{V_D}$  be  $v$ -full. For every  $\alpha, \beta \in \mathfrak{B}$ , the following statements hold:*

- (i)  $\alpha \rightarrow_{D \ominus v} \beta$  if and only if  $\alpha + v \rightarrow_{D \ominus v} \beta + v$ ;
- (ii) If  $\mathbb{p}(\alpha) \rightarrow_{D-v} \mathbb{p}(\beta)$ , then either  $\alpha \rightarrow_{D \ominus v} \beta$  or  $\alpha \rightarrow_{D \ominus v} \beta + v$ .
- (iii) If  $\mathbb{p}(\alpha) \rightarrow_{D-v} \mathbb{p}(\beta)$ , then either  $\alpha \rightarrow_{D \ominus v} \beta$  or  $\alpha + v \rightarrow_{D \ominus v} \beta$ .

*Proof.* It is a simple matter of routine checking. □

**Lemma 10.3.** *Let  $\mathfrak{B} \subseteq 2^{V_D}$  be  $v$ -full. Suppose that  $\mathbb{p}(\mathfrak{B}) \rightarrow_{D-v} \mathbb{p}(\mathfrak{B})$ . If  $\gamma \rightarrow_{D \ominus v} \gamma + v$  for a  $\gamma \in \mathfrak{B}$ , then  $\mathfrak{B} \rightarrow_{D \ominus v} \mathfrak{B}$ .*

*Proof.* For every  $\alpha, \beta \in \mathfrak{B}$ , one of the following four cases occurs according to Lemma 10.2:

- $\alpha \rightarrow_{D \ominus v} \gamma \rightarrow_{D \ominus v} \beta$ ;
- $\alpha \rightarrow_{D \ominus v} \gamma \rightarrow_{D \ominus v} \gamma + v \rightarrow_{D \ominus v} \beta$ ;
- $\alpha \rightarrow_{D \ominus v} \gamma + v \rightarrow_{D \ominus v} \beta$ ;
- $\alpha \rightarrow_{D \ominus v} \gamma + v \rightarrow_{D \ominus v} \gamma \rightarrow_{D \ominus v} \beta$ .

In all, they establish that  $\alpha \rightarrow_{D \ominus v} \beta$  and hence the lemma. □

**Lemma 10.4.** *Let  $\mathfrak{B} \subseteq 2^{V_D}$  be  $v$ -full. Suppose that  $\mathbb{p}(\mathfrak{B}) \rightarrow_{D-v} \mathbb{p}(\mathfrak{B})$ . If  $\mathcal{PS}(D \ominus v)[\mathfrak{B}]$  is weakly connected, then  $\gamma \rightarrow_{D \ominus v} \gamma + v$  for some  $\gamma \in \mathfrak{B}$ .*

*Proof.* Pick an  $\alpha \in \mathfrak{B}$ , and let  $\mathfrak{B}_0 = \{\beta \in \mathfrak{B} \mid \alpha \rightarrow_{D \ominus v} \beta\}$  and  $\mathfrak{B}_1 = \{\beta \in \mathfrak{B} \mid \alpha + v \rightarrow_{D \ominus v} \beta\}$ . It follows from Lemma 10.2(iii) that  $\mathfrak{B}_0 \cup \mathfrak{B}_1 = \mathfrak{B}$ . Since  $\mathcal{PS}_{D \ominus v}[\mathfrak{B}]$  is weakly connected, there exists an arc between  $\mathfrak{B}_0$  and  $\mathfrak{B}_1$ , say an arc  $\overrightarrow{\alpha\beta}$  from  $\mathfrak{B}_0$  to  $\mathfrak{B}_1$ . This implies that  $\beta \in \mathfrak{B}_0 \cap \mathfrak{B}_1$ . Making use of Lemma 10.2(ii), we get either  $\beta \rightarrow_{D \ominus v} \alpha + v$  or  $\beta \rightarrow_{D \ominus v} \alpha$ . Invoking the fact that  $\beta \in \mathfrak{B}_0 \cap \mathfrak{B}_1$ , we then come to either  $\alpha \rightarrow_{D \ominus v} \beta \rightarrow_{D \ominus v} \alpha + v$  or  $\alpha + v \rightarrow_{D \ominus v} \beta \rightarrow_{D \ominus v} \alpha$ , as was to be shown. □

**Lemma 10.5.** *Let  $\mathfrak{B} \subseteq 2^{V_D}$  be  $v$ -full. Suppose that  $\mathcal{PS}_{D-v}[\mathbb{p}(\mathfrak{B})]$  is weakly connected. If there exists a  $\gamma \in \mathfrak{B}$  such that  $\gamma \rightsquigarrow_{D \ominus v \parallel \mathfrak{B}} \gamma + v$ , then  $\mathcal{PS}_{D \ominus v}[\mathfrak{B}]$  is weakly connected.*

*Proof.* Taking an arbitrary  $\alpha \in \mathfrak{B}$ , our goal is to show that  $\alpha \rightsquigarrow_{D \ominus v \parallel \mathfrak{B}} \gamma$ .

As  $\mathcal{PS}_{D-v}[\mathbb{p}(\mathfrak{B})]$  is weakly connected, there exists  $W \in V_{D-v}^*$  such that  $\mathbb{p}(\alpha) \xrightarrow{W}_{D-v} \mathbb{p}(\gamma)$ . This gives either  $\alpha \xrightarrow{W}_{D \ominus v \parallel \mathfrak{B}} \gamma$  or  $\alpha \xrightarrow{W}_{D \ominus v \parallel \mathfrak{B}} \gamma + v$ . In view of  $\gamma + v \rightsquigarrow_{D \ominus v \parallel \mathfrak{B}} \gamma$ , the proof is thus furnished. □

**Lemma 10.6.** *Let  $\mathfrak{B} \subseteq 2^{V_D}$  be  $v$ -full. Suppose that  $\mathcal{PS}_{D-v}[\mathbb{p}(\mathfrak{B})]$  is strongly connected. If there exists a  $\gamma \in \mathfrak{B}$  such that  $\gamma \rightsquigarrow_{D \ominus v \parallel \mathfrak{B}} \gamma + v$ , then  $\mathfrak{B} \rightarrow_{D \ominus v} \mathfrak{B}$ .*

*Proof.* Follows from a combination of Lemmas 10.3, 10.4 and 10.5. □

For a word  $W = w_1 w_2 \dots w_k \in V_D^*$ , let  $\Lambda_W$  stand for the set  $\{\sum_{i=1}^j N_{D \ominus v}(w_i) \mid 0 \leq j \leq k\}$ .

**Lemma 10.7.** *Let  $\mathfrak{B}$  be a coset of  $\mathfrak{C}(D \ominus v)$  and  $C$  a subset of  $\mathfrak{B}$ . Set  $\overline{C} = \mathfrak{B} \setminus C$ . Suppose that  $W = w_1 \dots w_t \in L(D \ominus v)^*$  is a word such that  $\sum_{i=1}^t N_{D \ominus v}(w_i) = v$ . If  $|\mathfrak{C}_{D-v}| > |\mathbb{p}(\Lambda_W) + \mathbb{p}(C)|$ , then there exists  $\gamma \in \overline{C}$  such that  $\gamma \xrightarrow{W}_{D \ominus v \parallel \overline{C}} \gamma + v$ .*

*Proof.* Firstly, for every  $\gamma \in 2^{V_D}$ , it holds  $\mathbb{p}(\gamma + \Lambda_W) \cap \mathbb{p}(C) = \emptyset$  if and only if  $\mathbb{p}(\gamma) \notin \mathbb{p}(\Lambda_W) + \mathbb{p}(C)$ . Since  $|\mathbb{p}(\Lambda_W) + \mathbb{p}(C)| < |\mathfrak{C}_{D-v}| = |\mathbb{p}(\mathfrak{B})|$ , we can thus pick  $\gamma \in \mathfrak{B}$  satisfying  $\mathbb{p}(\gamma + \Lambda_W) \cap \mathbb{p}(C) = \emptyset$ . It is now no hard to see that  $\gamma \xrightarrow{W}_{D \ominus v \parallel \overline{C}} \gamma + \chi v$ . □

**Lemma 10.8.** *Let  $\mathfrak{B}$  be a coset of  $\mathfrak{C}_{D \ominus v}$  and  $C = \{0, v, L_{D \ominus v}, L_{D \ominus v} + v\}$ . Set  $\overline{C} = \mathfrak{B} \setminus C$ . Assume that  $\dim \mathfrak{C}_{D-v} \geq 3$ . Let  $W = w_1 \dots w_t \in V_{D-v}^*$  be a word satisfying  $\sum_{i=1}^t N_{D \ominus v}(w_i) = v$ . There exists  $\gamma \in \overline{C}$  such that  $\gamma \rightsquigarrow_{D \ominus v \parallel \overline{C}} \gamma + v$  provided that one of the following conditions holds:*

- (1)  $t = 4$ ,  $W \in L_{D \ominus v}^*$  and  $\{L_{D \ominus v}, L_{D \ominus v} + v\} \cap \Lambda_W \neq \emptyset$ ;
- (2)  $t \leq 3$  and  $W \in L_{D \ominus v}^*$ ;
- (3)  $t = 3$ ,  $\{i \mid w_i \in L_{D \ominus v}^*\} = 1$  and the only element from  $\{w_1, w_2, w_3\} \setminus L_{D \ominus v}$  is not a source in  $D \ominus v$ .

10.5. From  $D \ominus v$  to  $D$

10.6. From  $D \boxplus v$  to  $D$ , Gaussian elimination

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